

PAPPUS THEOREM, SCHWARTZ REPRESENTATIONS AND ANOSOV REPRESENTATIONS

THIERRY BARBOT, GYE-SEON LEE, AND VIVIANE PARDINI VALÉRIO

ABSTRACT. In the paper *Pappus's theorem and the modular group* [13], R. Schwartz constructed a 2-dimensional family of faithful representations ρ_Θ of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ into the group \mathcal{G} of projective symmetries of the projective plane via Pappus Theorem. If $\mathrm{PSL}(2, \mathbb{Z})_o$ denotes the unique index 2 subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ and $\mathrm{PGL}(3, \mathbb{R})$ the subgroup of \mathcal{G} consisting of projective transformations, then the image of $\mathrm{PSL}(2, \mathbb{Z})_o$ under ρ_Θ is in $\mathrm{PGL}(3, \mathbb{R})$. The representations ρ_Θ share a very interesting property with Anosov representations of surface groups into $\mathrm{PGL}(3, \mathbb{R})$: It preserves a topological circle in the flag variety. However, the representation ρ_Θ itself cannot be Anosov since the Gromov boundary of $\mathrm{PSL}(2, \mathbb{Z})$ is a Cantor set and not a circle.

In her PhD Thesis [15], V. P. Valério elucidated the Anosov-like feature of the Schwartz representations by showing that for each representation ρ_Θ , there exists an 1-dimensional family of representations $(\rho_\Theta^\varepsilon)_{\varepsilon \in \mathbb{R}}$ of $\mathrm{PSL}(2, \mathbb{Z})_o$ into $\mathrm{PGL}(3, \mathbb{R})$ such that ρ_Θ^0 is the restriction of the Schwartz representation ρ_Θ to $\mathrm{PSL}(2, \mathbb{Z})_o$ and ρ_Θ^ε is Anosov for every $\varepsilon < 0$. This result was announced and presented in her paper [14]. In the present paper, we extend and improve her work. For every representation ρ_Θ , we build a 2-dimensional family of representations $(\rho_\Theta^\lambda)_{\lambda \in \mathbb{R}^2}$ of $\mathrm{PSL}(2, \mathbb{Z})_o$ into $\mathrm{PGL}(3, \mathbb{R})$ such that $\rho_\Theta^\lambda = \rho_\Theta^\varepsilon$ for $\lambda = (\varepsilon, 0)$ and ρ_Θ^λ is Anosov for every $\lambda \in \mathcal{R}^\circ$, where \mathcal{R}° is an open set of \mathbb{R}^2 containing $\{(\varepsilon, 0) \mid \varepsilon < 0\}$. Moreover, among the 2-dimensional family of new Anosov representations, an 1-dimensional subfamily of representations can extend to representations of $\mathrm{PSL}(2, \mathbb{Z})$ into \mathcal{G} , and therefore the Schwartz representations are, in a sense, on the boundary of the Anosov representations in the space of all representations of $\mathrm{PSL}(2, \mathbb{Z})$ into \mathcal{G} .

1. INTRODUCTION

The initial goal of this work is to understand the similarity between Schwartz representations ρ_Θ of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ into the group \mathcal{G} of projective symmetries, presented in Schwartz [13, Theorem 2.4], and Anosov representations of Gromov-hyperbolic groups, which were studied by Labourie [9] and Guichard–Wienhard [6].

The starting point is a classical theorem due to Pappus of Alexandria (290 AD - 350 AD) known as Pappus's (hexagon) theorem (see Figure 1). As said by Schwartz, a slight twist makes this old theorem new again. This twist is to iterate, and thereby Pappus Theorem becomes a dynamical system. An important insight of Schwartz was to describe this dynamic through objects named by him marked boxes. A marked box $[\Theta]$ is simply a collection of points and lines in the projective plane $\mathbf{P}(V)$ obeying certain rules (see Section 2.2). When the Pappus theorem is applied to a marked box, more points and lines are produced, and so on.

The dynamics on the set \mathcal{CM} of marked boxes $[\Theta]$ come from the actions of two special groups \mathcal{G} and \mathfrak{G} . The group \mathcal{G} of projective symmetries is the group of transformations of the flag variety \mathcal{F} , i.e. the group \mathcal{G} makes up of the projective transformations (i.e. collineations)

Key words and phrases. Pappus Theorem, modular group, group of projective symmetries, Farey triangulation, Schwartz representation, Gromov-hyperbolic group, Anosov representations, Hilbert metric.

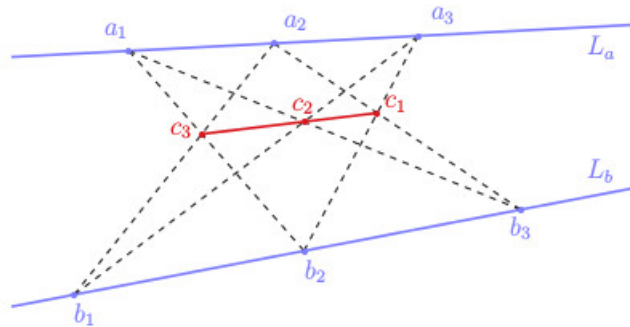


FIGURE 1. **Pappus Theorem:** If the points a_1, a_2, a_3 are colinear and the points b_1, b_2, b_3 are colinear, then the points c_1, c_2, c_3 are also colinear.

and dualities (i.e. correlations). The action of \mathcal{G} on \mathcal{CM} is essentially given by the fact that a marked box is characterized by a collection of flags in \mathcal{F} . The group \mathfrak{G} of elementary transformations of marked boxes is generated by a natural involution i , and transformations τ_1, τ_2 induced by Pappus Theorem (see Section 3.4). The group \mathfrak{G} is isomorphic to the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

Another Schwartz insight was that for each convex marked box $[\Theta]$, there is another action of the modular group on the \mathfrak{G} -orbit of $[\Theta]$, commuting with the action of \mathfrak{G} . This action can be described in the following way: On the one hand, the isometric action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ on the hyperbolic plane \mathbb{H}^2 preserves the set \mathcal{L}_o of Farey geodesics. On the other hand, there is a natural labeling on \mathcal{L}_o by the elements of the \mathfrak{G} -orbit of $[\Theta]$. Hence the action of $\mathrm{PSL}(2, \mathbb{Z})$ on labels induces an action on the \mathfrak{G} -orbit of $[\Theta]$. Moreover, this labeling allows us to better understand how the elements of the \mathfrak{G} -orbit of $[\Theta]$ are nested when viewed in the projective plane $\mathbf{P}(V)$ (or when viewed in the dual projective plane $\mathbf{P}(V^*)$).

Through these two actions on \mathcal{CM} , Schwartz showed that for each convex marked box $[\Theta]$, there is a faithful representation $\rho_\Theta : \mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathcal{G}$ such that for every γ in $\mathrm{PSL}(2, \mathbb{Z})$ and every Farey geodesic $e \in \mathcal{L}_o$, the label of $\gamma(e)$ is the image of the label of e under $\rho_\Theta(\gamma)$ (see Theorem 4.7).

As observed in Barbot [1, Remark 5.13], the Schwartz representations ρ_Θ , in their dynamical behavior, look like *Anosov representations*, introduced by Labourie [9] in order to study the Hitchin component of the space of representations of closed surface groups. Posteriorly, Guichard and Wienhard [6] enlarged this concept to the framework of Gromov-hyperbolic groups, which allows us to define the notion of Anosov representations of $\mathrm{PSL}(2, \mathbb{Z})$. Anosov representations currently play an important role in the development of higher Teichmüller theory (see e.g. Bridgeman–Canary–Labourie–Sambarino [3]).

In this paper, we show that Schwartz representations are not Anosov, but limits of Anosov representations. More precisely:

Theorem 1.1. *Let $[\Theta]$ be a convex marked box. Then there is a two-dimensional family of representations*

$$\rho_\Theta^\lambda : \mathrm{PSL}(2, \mathbb{Z})_o \rightarrow \mathrm{PGL}(3, \mathbb{R}),$$

where λ is a pair of real parameters (ε, δ) and $\mathrm{PSL}(2, \mathbb{Z})_o$ is the unique subgroup of index 2 in $\mathrm{PSL}(2, \mathbb{Z})$, such that:

- (1) If $\lambda = (0, 0)$, then ρ_Θ^λ coincides with the restriction of the Schwartz representation ρ_Θ to $\mathrm{PSL}(2, \mathbb{Z})_o$.

- (2) If $\lambda \in \mathcal{R}$, then ρ_Θ^λ is faithful (see Equation (4) and Figure 11 for the description of the region \mathcal{R} of \mathbb{R}^2).
- (3) If $\lambda \in \mathcal{R}^\circ$, then ρ_Θ^λ is Anosov.

Finally, by understanding the extension of the representations ρ_Θ^λ to $\mathrm{PSL}(2, \mathbb{Z})$, we can prove the following:

Theorem 1.2. *Let $[\Theta]$ be a convex marked box, and let $\rho_\Theta^\lambda : \mathrm{PSL}(2, \mathbb{Z})_o \rightarrow \mathrm{PGL}(3, \mathbb{R})$ be the representations as in Theorem 1.1. Then there exist a real number $\varepsilon_0 < 0$ and a function $\delta_h :]\varepsilon_0, 0] \rightarrow \mathbb{R}$ such that for $\lambda = (\varepsilon, \delta_h(\varepsilon))$,*

- (1) *If $\varepsilon \in]\varepsilon_0, 0]$, then ρ_Θ^λ extends naturally to a representation $\bar{\rho}_\Theta^\lambda$ of Γ into \mathcal{G} .*
- (2) *If $\varepsilon = 0$, then $\bar{\rho}_\Theta^\lambda = \rho_\Theta$.*
- (3) *If $\varepsilon \in]\varepsilon_0, 0[$, then ρ_Θ^λ is Anosov.*

The remainder of this paper is organized as follows.

In Section 2, we describe the dynamics on marked boxes generated by Pappus Theorem. In Section 3, we introduce the group \mathcal{G} of projective symmetries and the group \mathfrak{G} of elementary transformations of marked boxes. In Section 4, we present Schwartz representations, which involves a labeling on Farey geodesics by the orbit of a marked box under \mathfrak{G} . In Section 5, we define Anosov representations. After that, we start the original content of this paper. In Section 6, we construct our new elementary transformations on marked boxes and our new representations. In Section 7, we explain how to define special norms on the projective plane (or its dual plane) for each convex marked box, and using them, in Section 8, we prove that our new representations are Anosov, which establish Theorem 1.1. The last Section 9 is devoted to understand how to extend new representations to $\mathrm{PSL}(2, \mathbb{Z})$ for the proof of Theorem 1.2.

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2. A DYNAMIC OF PAPPUS THEOREM VIA MARKED BOXES

As in Schwartz [13], we consider the Pappus Theorem as a dynamical system defined on objects called marked boxes. A marked box is essentially a collection of points and lines in the projective plane satisfying the rules that we present below.

2.1. Pappus Theorem. Let V be a 3-dimensional real vector space and let $\mathbf{P}(V)$ be the projective space associated to V , i.e. the space of 1-dimensional subspaces of V . If a and b are two distinct points of $\mathbf{P}(V)$, then ab denotes the line through a and b . In a similar way, if A and B are two distinct lines of $\mathbf{P}(V)$, then AB denotes the intersection point of A and B .

Theorem 2.1 (Pappus Theorem). *If the points a_1, a_2, a_3 are colinear and the points b_1, b_2, b_3 are colinear in $\mathbf{P}(V)$, then the points $c_1 = (a_2b_3)(a_3b_2)$, $c_2 = (a_3b_1)(a_1b_3)$, $c_3 = (a_1b_2)(a_2b_1)$ are also colinear in $\mathbf{P}(V)$.*

We say that the Pappus Theorem is on *generic conditions* if a_1, a_2, a_3 are distinct points of a line L_a , as well as b_1, b_2, b_3 are distinct points of a line L_b , and $a_i \notin L_b, b_i \notin L_a$ for all $i = 1, 2, 3$. When the Pappus Theorem is on generic conditions, we have a *Pappus configuration* formed by the points $a_1, a_2, a_3, b_1, b_2, b_3$. An important fact is that the Pappus Theorem on generic conditions can be iterated infinitely many times (see Figure 2), i.e. a Pappus configuration is stable, and therefore it gives us a dynamical system.

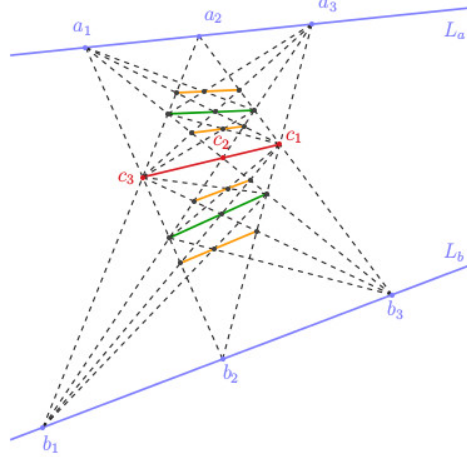


FIGURE 2. An iteration of the Pappus Theorem

2.2. Marked boxes. Let V^* be the dual vector space of V and let $\mathbf{P}(V^*)$ be the projective space associated to V^* , i.e. the space of lines of $\mathbf{P}(V)$. An *overmarked box* Θ of $\mathbf{P}(V)$ is a pair of distinct 6 tuples having the incidence relations shown in Figure 3:

$$\begin{aligned} \Theta &= ((p, q, r, s, t, b), (P, Q, R, S, T, B)) \\ p, q, r, s, t, b &\in \mathbf{P}(V) \text{ and } P, Q, R, S, T, B \in \mathbf{P}(V^*) \\ TB \notin \{p, q, r, s, t, b\}, P &= ts, Q = tr, R = bq, S = bp, T = pq \text{ and } B = rs \end{aligned}$$

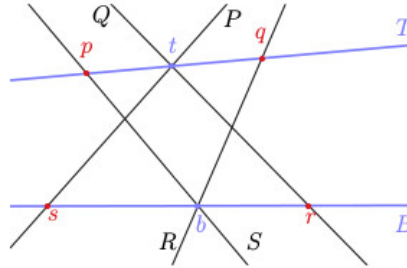


FIGURE 3. An overmarked box in $\mathbf{P}(V)$

The overmarked box Θ is completely determined by the 6 tuple (p, q, r, s, t, b) , but it is wise to keep in mind that we should treat equally the dual counterpart (P, Q, R, S, T, B) . The

dual of Θ , denoted by Θ^* , is $((P, Q, R, S; T, B), (p, q, r, s; t, b))$. The *top* (flag) of Θ is the pair (t, T) and the *bottom* (flag) of Θ is the pair (b, B) .

We denote the set of overmarked boxes by \mathcal{CSM} . Let $j : \mathcal{CSM} \rightarrow \mathcal{CSM}$ be the involution given by

$$((p, q, r, s; t, b), (P, Q, R, S; T, B)) \mapsto ((q, p, s, r; t, b), (Q, P, S, R; T, B)).$$

A *marked box* is an equivalence class of overmarked boxes under this involution j . We denote the set of marked boxes by \mathcal{CM} . A overmarked box $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$ (or a marked box $[\Theta]$) is *convex* if the following two conditions hold:

- The points p and q separate t and TB on the line T .
- The points r and s separate b and TB on the line B .

Given a marked box $[\Theta]$, we can define the segments $[pq]$ (resp. $[rs]$) as the closure of the complement in T (resp. B) of $\{p, q\}$ (resp. $\{r, s\}$) containing t (resp. b). There are three ways to choose the segments $[qr]$, $[sp]$ simultaneously so that they do not intersect. The marked box $[\Theta]$ is convex if and only if one of these three choices leads to a quadrilateral (p, q, r, s) (in this cyclic order) the boundary of which is not freely homotopic to a line of $\mathbf{P}(V)$. We then define the *convex interior*, denoted by $[\Theta]^\circ$, of the convex marked box $[\Theta]$ as the interior of the convex quadrilateral (p, q, r, s) in $\mathbf{P}(V)$ (see Figure 4).

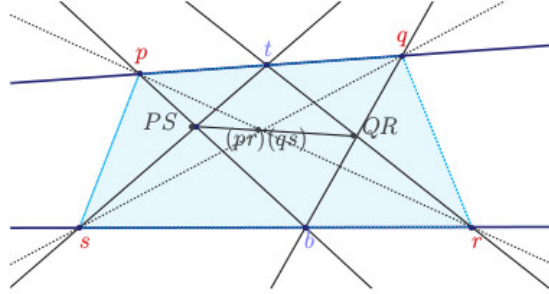


FIGURE 4. A convex interior $[\Theta]^\circ$ of $[\Theta]$ in $\mathbf{P}(V)$ is drawn in blue.

Finally, a marked box $[\Theta]$ is convex if and only if the dual $[\Theta^*]$ of $[\Theta]$ is convex, and in this case, we denote by $[\Theta^*]^\circ$ the convex interior of $[\Theta^*]$ in $\mathbf{P}(V^*)$. Be careful that $[\Theta^*]^\circ$ is *not* the convex domain dual to $[\Theta]^\circ$.

3. TWO GROUPS ACTING ON MARKED BOXES

Following Schwartz [13], we will explain how the group of projective symmetries acts on marked boxes, and introduce the group of elementary transformations of marked boxes.

3.1. The group \mathcal{G} of projective symmetries. Recall that V is a 3-dimensional real vector space and V^* is its dual vector space. We denote by $\langle v^* | v \rangle$ the evaluation of an element v^* of V^* on an element v of V . If W is a vector space and $f : V \rightarrow W$ is a linear isomorphism between V and W , then the dual map of f is the linear isomorphism $f^* : W^* \rightarrow V^*$ such that $\langle f^*(w^*) | v \rangle = \langle w^* | f(v) \rangle$ for all $w^* \in W^*$ and $v \in V$.

The projectivization of $f : V \rightarrow W$ is denoted by $\mathbf{P}(f) : \mathbf{P}(V) \rightarrow \mathbf{P}(W)$. A *projective transformation* T of $\mathbf{P}(V)$ is a transformation of $\mathbf{P}(V)$ induced by an automorphism g of V , i.e. $T = \mathbf{P}(g)$, and the dual map T^* of T is the transformation $\mathbf{P}(g^*)^{-1}$ of $\mathbf{P}(V^*)$. A *projective duality* D is a homeomorphism between $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ induced by an isomorphism h

between V and V^* , i.e. $D = \mathbf{P}(h)$, and the dual map D^* of D is the homeomorphism $(\mathbf{P}(h^*) \circ \mathbf{P}(I))^{-1} : \mathbf{P}(V^*) \rightarrow \mathbf{P}(V)$, where $I : V \rightarrow V^{**}$ is the canonical linear isomorphism between V and V^{**} . We denote by $[v]$ the 1-dimensional subspace spanned by a non-zero v of V . The *flag variety* \mathcal{F} is the subset of $\mathbf{P}(V) \times \mathbf{P}(V^*)$ formed by all pairs $([v], [v^*])$ satisfying $\langle v^* | v \rangle = 0$.

If T is a projective transformation of $\mathbf{P}(V)$, then there is an automorphism $\mathcal{T} : \mathcal{F} \rightarrow \mathcal{F}$, called also *projective transformation*, defined by

$$(1) \quad \mathcal{T}(x, X) = (T(x), T^*(X)) \quad \text{for every } (x, X) \in \mathcal{F}.$$

Similarly, if $D : \mathbf{P}(V) \rightarrow \mathbf{P}(V^*)$ is a duality, then there is an automorphism $\mathcal{D} : \mathcal{F} \rightarrow \mathcal{F}$, also called *duality*, defined by

$$(2) \quad \mathcal{D}(x, X) = (D^*(X), D(x)) \quad \text{for every } (x, X) \in \mathcal{F}.$$

Let \mathcal{H} be the set of projective transformations of \mathcal{F} as in (1), and let \mathcal{G} be the set formed by \mathcal{H} and dualities of \mathcal{F} as in (2). This set \mathcal{G} is the *group of projective symmetries* with the obvious composition operation. The subgroup \mathcal{H} of \mathcal{G} has index 2.

Remark 3.1. If we equip V with a basis \mathcal{B} , and V^* with the dual basis \mathcal{B}^* , then the projective space $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ can be identified with $\mathbf{P}(\mathbb{R}^3)$. A duality $D : \mathbf{P}(V) \rightarrow \mathbf{P}(V^*)$ is given by a unique element $A \in \text{PGL}(3, \mathbb{R})$, and the flag transformation \mathcal{D} is expressed by the map

$$(x, X) \mapsto ({}^tA^{-1}X, Ax)$$

from $\{(x, X) \in \mathbf{P}(\mathbb{R}^3) \times \mathbf{P}(\mathbb{R}^3) \mid x \cdot X = 0\}$ into itself, where tA denotes the transpose of A and $x \cdot X$ is the dot product of x and X . It follows that involutions in $\mathcal{G} \setminus \mathcal{H}$ correspond to dualities D for which A is symmetric. They are precisely *polarities*, i.e. isomorphisms h between V and V^* for which $(u, v) \mapsto \langle h(u) | v \rangle$ is a non-degenerate symmetric bilinear form.

3.2. The action of \mathcal{G} on marked boxes. Consider the map $\Upsilon : \mathcal{CSM} \rightarrow \mathcal{F}^6$ defined by:

$$\Upsilon((p, q, r, s; t, b), (P, Q, R, S; T, B)) = ((t, T), (t, P), (t, Q), (b, B), (b, R), (b, S))$$

It is a bijection onto some subset of \mathcal{F}^6 (which is not useful to describe further). It induces a map from \mathcal{CSM} into the quotient of \mathcal{F}^6 by the involution permuting the second and the third factor, and the fifth and the sixth factor, and hence it gives us a natural action of the group of projective symmetries on \mathcal{CM} . However, as Schwartz observed, it is *not* the one we should consider because with this choice the Schwartz Representation Theorem (Theorem 4.7) would fail.

If T is a projective transformation of $\mathbf{P}(V)$ that induces $\mathcal{T} \in \mathcal{H} \subset \mathcal{G}$, then we define a map $\mathcal{T} : \mathcal{CSM} \rightarrow \mathcal{CSM}$ by

$$\mathcal{T}(\Theta) = ((\hat{p}, \hat{q}, \hat{r}, \hat{s}; \hat{t}, \hat{b}), (\hat{P}, \hat{Q}, \hat{R}, \hat{S}; \hat{T}, \hat{B})) \quad \text{for every } \Theta \in \mathcal{CSM},$$

where $\hat{x} = T(x)$ for $x \in \mathbf{P}(V)$ and $\hat{X} = T^*(X)$ for $X \in \mathbf{P}(V^*)$.

If D is a duality that induces $\mathcal{D} \in \mathcal{G} \setminus \mathcal{H}$, then we define a map $\mathcal{D} : \mathcal{CSM} \rightarrow \mathcal{CSM}$ (notice the [Schwartz reordering](#)) by

$$\mathcal{D}(\Theta) = ((P^*, Q^*, S^*, R^*; T^*, B^*), (q^*, p^*, r^*, s^*; t^*, b^*)) \quad \text{for every } \Theta \in \mathcal{CSM},$$

where $X^* = D^*(X)$ for $X \in \mathbf{P}(V^*)$ and $x^* = D(x)$ for $x \in \mathbf{P}(V)$.

It is clear that both transformations \mathcal{T} and \mathcal{D} commute with the involution j , and so it induces an action of \mathcal{G} on \mathcal{CM} , which furthermore preserves the convexity of marked boxes. We will see in Section 3.4 that this action commutes with elementary transformations of marked boxes.

Remark 3.2. Given two dualities \mathcal{D}_1 and \mathcal{D}_2 belonging to $\mathcal{G} \setminus \mathcal{H}$, we have

$$\mathcal{D}_2(\mathcal{D}_1(\Theta)) = j((\mathcal{D}_2\mathcal{D}_1)(\Theta))$$

where $j : \mathcal{CSM} \rightarrow \mathcal{CSM}$ is the involution defining marked boxes. Hence, the action of \mathcal{G} on \mathcal{CM} defined by Schwartz does not lift to an action of \mathcal{G} on \mathcal{CSM} .

3.3. The space of marked boxes modulo \mathcal{G} . Let

$$\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$$

be a convex overmarked box. There is a unique basis (up to scaling), called the Θ -basis, of V for which the points p, q, r, s of Θ have the projective coordinates:

$$p = [-1 : 1 : 0], \quad q = [1 : 1 : 0], \quad r = [1 : 0 : 1], \quad s = [-1 : 0 : 1]$$

The convexity of Θ implies that for the Θ -basis of V , the points t and b have the projective coordinates:

$$t = [\zeta_t : 1 : 0] \text{ and } b = [\zeta_b : 0 : 1] \text{ with some } \zeta_t, \zeta_b \in]-1, 1[$$

(see Figure 10). In other words, the convex overmarked box Θ is characterized, modulo \mathcal{H} , by the coordinates ζ_t and ζ_b of respectively t, b for the Θ -basis of V , and the space of overmarked boxes modulo \mathcal{H} is therefore $] -1, 1[^2$. We call Θ a convex (ζ_t, ζ_b) -overmarked box. When $(\zeta_t, \zeta_b) = (0, 0)$, the overmarked box is said to be *special*.

The involution j maps a (ζ_t, ζ_b) -overmarked box to a $(-\zeta_t, -\zeta_b)$ -overmarked box, and hence the set of marked boxes modulo \mathcal{H} is the quotient of $] -1, 1[^2$ by the involution $(\zeta_t, \zeta_b) \mapsto (-\zeta_t, -\zeta_b)$. Moreover, the special marked boxes are the only marked boxes that can be preserved by a non-trivial projective transformation, and for each special marked box $[\Theta]$, there is only one such projective transformation \mathcal{T} fixing $[\Theta]$, which satisfies $\mathcal{T}(\Theta) = j(\Theta)$.

Assume now that some duality $D : \mathbf{P}(V) \rightarrow \mathbf{P}(V^*)$ induces a duality \mathcal{D} fixing the marked box $[\Theta]$. At the level of overmarked boxes, we have either $\mathcal{D}(\Theta) = \Theta$ or $\mathcal{D}(\Theta) = j(\Theta)$. In both cases, since j and \mathcal{D} commute, we have $\mathcal{D}(\mathcal{D}(\Theta)) = \Theta$. According to Remark 3.2, the projective symmetry \mathcal{D}^2 is a projective transformation mapping Θ onto $j(\Theta)$, hence Θ is special. In this case, if we equip V with the Θ -basis of V , and V^* with its dual basis, then there is a unique $A \in \text{GL}(3, \mathbb{R})$ (up to scaling) that induces the duality $D : \mathbf{P}(V) \rightarrow \mathbf{P}(V^*)$, and the matrix A satisfies the following (see Remark 3.1):

$${}^tA^{-1}A = \mu \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \mu \in \mathbb{R}^*$$

Since ${}^tA^{-1}A$ has determinant 1, the scaling factor μ is -1 , and it follows that, up to rescaling:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \nu \\ 0 & -\nu & 0 \end{pmatrix}$$

with $\nu \in \mathbb{R}^*$. Now, such a matrix corresponds to a duality sending Θ onto $j(\Theta)$ if and only if $\nu = 1$, and sending Θ onto Θ if and only if $\nu = -1$.

In summary, we prove:

Proposition 3.3. The space of convex marked boxes up to projective symmetries is the quotient of $] -1, 1[^2 \setminus \{0\}$ by $\{\pm \text{Id}\}$, to which is added a singular point corresponding to special marked boxes. The \mathcal{G} -stabilizer of a marked box $[\Theta]$ is of order 2 without a duality if Θ is not special, and is a cyclic group of order 4 generated by a duality which is not a polarity if $[\Theta]$ is special. \square

3.4. The group \mathfrak{G} of elementary transformations of marked boxes. Let

$$\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B)) \in \mathcal{CSM}.$$

The Pappus Theorem gives us two new elements of \mathcal{CSM} that are images of Θ under two special permutations τ_1 and τ_2 on \mathcal{CSM} (see Figure 5). These permutations are defined by:

$$\tau_1(\Theta) = ((p, q, QR, PS; t, (qs)(pr)), (P, Q, qs, pr; T, (QR)(PS)))$$

$$\tau_2(\Theta) = ((QR, PS, s, r; (qs)(pr), b), (pr, qs, S, R; (QR)(PS), B))$$

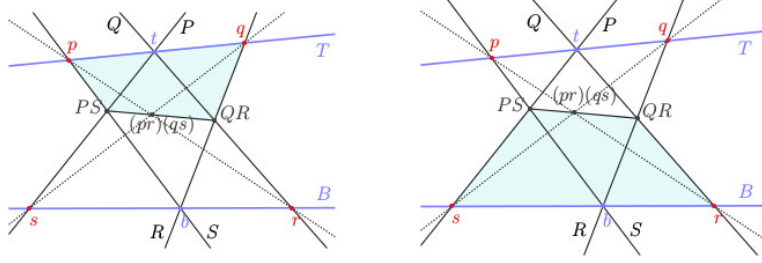


FIGURE 5. Two permutations τ_1 and τ_2 ; the convex interiors of $\tau_1(\Theta)$ and $\tau_2(\Theta)$ are drawn in blue when Θ is convex.

There is also a natural involution, denoted by i , on \mathcal{CSM} (see Figure 6) given by:

$$i(\Theta) = ((s, r, p, q; b, t), (R, S, Q, P; B, T))$$

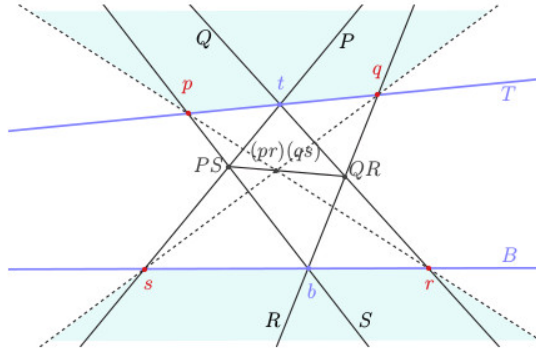


FIGURE 6. The permutation i ; the convex interior of $i(\Theta)$ is drawn in blue when Θ is convex.

The transformations i , τ_1 and τ_2 are permutations on \mathcal{CSM} commuting with j , hence they also act on \mathcal{CM} . We denote by $\mathcal{S}(\mathcal{CM})$ the group of permutations on \mathcal{CM} .

Remark 3.4. If $[\Theta]$ is convex, then the new boxes $[i(\Theta)]$, $[\tau_1(\Theta)]$ and $[\tau_2(\Theta)]$ are also convex. Moreover, $[\tau_1(\Theta)]^\circ \subsetneq [\Theta]^\circ$ and $[\tau_2(\Theta)]^\circ \subsetneq [\Theta]^\circ$, and therefore the semigroup \mathfrak{F} of $\mathcal{S}(\mathcal{CM})$ generated by τ_1 and τ_2 is free. On the other hand, $[i(\Theta)]^\circ \cap [\Theta]^\circ = \emptyset$. The convex interiors of these marked boxes are highlighted in Figures 5 and 6.

Remark 3.5. In the dual projective plane $\mathbf{P}(V^*)$, the inclusions are reversed:

$$[\tau_1(\Theta)^*]^\circ \supsetneq [\Theta^*]^\circ \quad \text{and} \quad [\tau_2(\Theta)^*]^\circ \supsetneq [\Theta^*]^\circ$$

However, we still have $[i(\Theta)^*]^\circ \cap [\Theta^*]^\circ = \emptyset$.

The permutations i , τ_1 and τ_2 on \mathcal{CM} are called *elementary transformations of marked boxes*. These transformations can be applied iteratively on the elements of \mathcal{CM} , so i , τ_1 and τ_2 generate a semigroup \mathfrak{S} of $\mathcal{S}(\mathcal{CM})$.

Lemma 3.6. The following relations hold:

$$i^2 = 1, \quad \tau_1 i \tau_2 = i, \quad \tau_2 i \tau_1 = i, \quad \tau_1 i \tau_1 = \tau_2, \quad \tau_2 i \tau_2 = \tau_1$$

Proof. See the proof in Schwartz [13, Lemma 2.3]. \square

Thus, by Lemma 3.6, the inverses of i , τ_1 and τ_2 in $\mathcal{S}(\mathcal{CM})$ are:

$$i^{-1} = i, \quad \tau_1^{-1} = i \tau_2 i, \quad \tau_2^{-1} = i \tau_1 i$$

Therefore the semigroup \mathfrak{S} is in fact a group, and this group \mathfrak{S} is called the *group of elementary transformations of marked boxes*.

Remark 3.7. The relations in Lemma 3.6 are true only when these transformations are applied on elements of \mathcal{CM} , and they are not valid on \mathcal{CSM} .

Remark 3.8. The action of \mathfrak{S} and \mathcal{G} on \mathcal{CM} commute each other.

Lemma 3.9. The action of the group \mathfrak{S} on the set of convex marked boxes is free.

Proof. Every element of \mathfrak{S} can be written as the form $i^a w i^b$, where $a, b \in \{0, 1\}$ and w is an element of the semigroup generated by τ_1 and τ_2 . Assume that $i^a w i^b [\Theta] = [\Theta]$ for some convex marked box $[\Theta]$ and a non-trivial element $i^a w i^b$ of \mathfrak{S} . Since $i[\Theta]^\circ \cap [\Theta]^\circ = \emptyset$, the element w is not trivial. By replacing $[\Theta]$ by $i[\Theta]$, we can further assume that $a = 0$. Then we have two cases: either $b = 0$ or $b = 1$. If $b = 0$, then $w[\Theta] = [\Theta]$, which is impossible since $w[\Theta]^\circ \subsetneq [\Theta]^\circ$ by Remark 3.4. If $b = 1$, then $w i[\Theta]^\circ$ is contained in $i[\Theta]^\circ$, therefore disjoint from $[\Theta]^\circ$, which is a contradiction. \square

It follows from Lemma 3.9 that if $\varrho_1 = i \tau_1$, then \mathfrak{S} admits the following group presentation:

$$\mathfrak{S} = \langle i, \varrho_1 \mid i^2 = 1, \varrho_1^3 = 1 \rangle$$

In particular, it is isomorphic to the modular group $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

4. SCHWARTZ REPRESENTATIONS

4.1. Farey geodesics labeled by the \mathfrak{S} -orbit of a marked box. Schwartz [13] discovered that the combinatorial structure of the orbit of a marked box under \mathfrak{S} can be nicely described by the Farey graph and its associated geodesics of the hyperbolic plane \mathbb{H}^2 . Here we give a short overview of the Farey graph (see e.g. Katok–Ugarcovici [8] or Morier-Genoud–Ovsienko–Tabachnikov [11]).

If we consider the elements I and R of the modular group $\mathrm{PSL}(2, \mathbb{Z})$:

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

then we have the group presentation:

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle I, R \mid I^2 = 1, R^3 = 1 \rangle$$

Remark 4.1. It will be essential for us to deal with the subgroup $\mathrm{PSL}(2, \mathbb{Z})_o$ of $\mathrm{PSL}(2, \mathbb{Z})$ generated by R and IRI . It consists of the elements of $\mathrm{PSL}(2, \mathbb{Z})$ that can be written as a word with the letters I and R , with an even number of I . Moreover, it is in fact the unique index 2 subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ since every morphism $\mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathbb{Z}_2$ must vanish on R . Finally, we can also characterize $\mathrm{PSL}(2, \mathbb{Z})_o$ as the set of elements of $\mathrm{PSL}(2, \mathbb{Z})$ whose trace is an odd integer.

Let Δ_0 be the ideal geodesic triangle in the upper half plane \mathbb{H}^2 whose vertices are $0, 1, \infty$ in $\partial\mathbb{H}^2$. For any two points $x, y \in \partial\mathbb{H}^2$, we denote by $[x, y]$ a unique geodesic joining x and y . The isometry R of \mathbb{H}^2 is the rotation of order 3 whose center is the “center” of the triangle Δ_0 and that permutes $1, 0, \infty$ in this (clockwise) cyclic order. The isometry I of \mathbb{H}^2 is the rotation of order 2 whose center is the orthogonal projection of the “center” of Δ_0 on the geodesic $[\infty, 0]$ (see Figure 7).

The Farey graph is a directed graph such that its vertices are the rational numbers together with the infinity $\infty = 1/0$, two vertices p/q and p'/q' , written in reduced form, are connected by an edge $(p/q, p'/q')$ if and only if the equality $pq' - p'q = \pm 1$ holds, and two adjacent vertices are connected by exactly two oriented edges e and \bar{e} , where \bar{e} is the same as e except the orientation.

It is well-known that if we realize every oriented edge $(p/q, p'/q')$ by the oriented geodesic $[p/q, p'/q']$ in \mathbb{H}^2 joining p/q and p'/q' in $\partial\mathbb{H}^2$ (the initial point p/q is called the *tail* and the final point p'/q' is called the *head* of $[p/q, p'/q']$), then we obtain a triangulation of \mathbb{H}^2 (see Figure 7), called the *Farey triangulation*. Each oriented geodesic in \mathbb{H}^2 that realizes an edge of the Farey graph is called a *Farey geodesic* and we denote the set of Farey geodesics by \mathcal{L}_o .

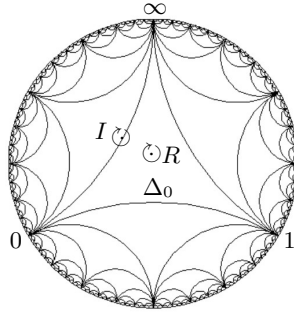


FIGURE 7. The Farey triangulation in the Poincaré disk model of \mathbb{H}^2

We can regard the Farey triangulation as a tiling: the Farey geodesics without orientation are edges of ideal geodesic triangles, called *Farey triangles*. For example, the triangle Δ_0 is a Farey triangle. The modular group $\mathrm{PSL}(2, \mathbb{Z}) \subset \mathrm{PSL}(2, \mathbb{R})$ acting on \mathbb{H}^2 preserves the Farey triangulation, and acts transitively on Farey triangles. Moreover, the stabilizer of Δ_0 in $\mathrm{PSL}(2, \mathbb{Z})$ is the subgroup of order 3 generated by R .

Remark 4.2. The index 2 subgroup $\mathrm{PSL}(2, \mathbb{Z})_o$ does not act transitively on Farey geodesics, but acts simply transitively on *non-oriented* Farey geodesics. In other way, once chosen a Farey geodesic e_0 (we will always take $e_0 = [\infty, 0]$), then $\mathrm{PSL}(2, \mathbb{Z})_o$ acts simply transitively on the orbit of e_0 under $\mathrm{PSL}(2, \mathbb{Z})_o$, which is called the $\mathrm{PSL}(2, \mathbb{Z})_o$ -orientation.

Now, there is another action of \mathfrak{S} , which is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$, on the set \mathcal{L}_o of Farey geodesics (but not really on the Farey graph since it does not respect the incidence relation of the graph) such that, for every Farey geodesic $e \in \mathcal{L}_o$, we have:

- $i(e)$ is the Farey geodesic \bar{e} , which is the same as e except the orientation,
- $\tau_1(e)$ is the Farey geodesic obtained by rotating e counterclockwise one “click” about its tail point,
- $\tau_2(e)$ is the Farey geodesic obtained by rotating e clockwise one “click” about its head point.

These actions of $\mathrm{PSL}(2, \mathbb{Z})$ and \mathfrak{G} on \mathcal{L}_o are both simply transitive and commute each other.

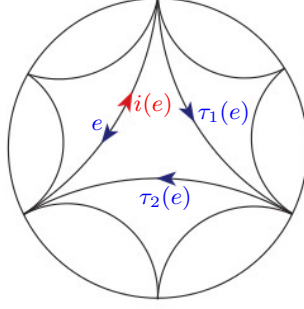


FIGURE 8. The action of \mathfrak{G} on the Farey geodesics

Remark 4.3. Note that an element σ of \mathfrak{G} preserves the $\mathrm{PSL}(2, \mathbb{Z})_o$ -orientation if and only if it belongs to the index 2 subgroup \mathfrak{G}_o made of elements that are products of an even number of generators i, τ_1, τ_2 . In other words, the $\mathrm{PSL}(2, \mathbb{Z})_o$ -orbits and the \mathfrak{G}_o -orbits coincide.

Given a convex marked box $[\Theta_0] \in \mathcal{CM}$, we can label each Farey geodesic e by an element $[\Theta](e)$ of the \mathfrak{G} -orbit of $[\Theta_0]$ as follows: first assign the label $[\Theta](e_0) = [\Theta_0]$ for the geodesic $e_0 = [\infty, 0]$, and then for every geodesic $e = \sigma e_0$ with $\sigma \in \mathfrak{G}$, define $[\Theta](e) = \sigma[\Theta_0]$. More generally, for any Farey geodesic $e \in \mathcal{L}_o$ and any Schwartz transformation $\sigma \in \mathfrak{G}$:

$$[\Theta](\sigma e) = \sigma[\Theta](e)$$

Remark 4.4. Using this labeling, we can easily see the nesting property of the marked boxes in the \mathfrak{G} -orbit of $[\Theta]$ viewed in $\mathbf{P}(V)$. For each oriented geodesic e , let H_e be the half space of \mathbb{H}^2 on the left of e . Assume that the label $[\Theta](e_0)$ of e_0 is convex. Let e, e' be two Farey geodesics. Then the following property is true: $H_{e'} \subset H_e$ if and only if the convex interior of $[\Theta](e')$ is contained in the convex interior of $[\Theta](e)$. In other words, $H_{e'} \subset H_e$ if and only if e' is obtained from e by applying a sequence of elementary transformations τ_1 and τ_2 . Moreover, e and e' have the same tail points (resp. head point) if and only if the marked boxes $[\Theta](e)$ and $[\Theta](e')$ have the same top (resp. bottom).

4.2. Construction of Schwartz representations. Now we explain how to build Schwartz representations.

Lemma 4.5. Let Θ be a convex overmarked box. Then

1. there is only one projective transformation $\mathcal{A}_\Theta^0 \in \mathcal{H}$ such that:

$$\Theta \xrightarrow{\mathcal{A}_\Theta^0} j\varrho_1\Theta \xrightarrow{\mathcal{A}_\Theta^0} \varrho_1^2\Theta \xrightarrow{\mathcal{A}_\Theta^0} j\Theta$$

2. there is only one duality $\mathcal{D}_\Theta^0 \in \mathcal{G} \setminus \mathcal{H}$ such that:

$$\Theta \xrightarrow{\mathcal{D}_\Theta^0} j(i\Theta) \xrightarrow{\mathcal{D}_\Theta^0} \Theta$$

Moreover, the duality \mathcal{D}_Θ^0 happens to be a polarity associated to a positive definite quadratic form (see Remark 3.1).

Proof. The proof is in Schwartz [13, Theorem 2.4] (see also Valério [15, Lemma 3.1] for more details). As usual, equip V with the Θ -basis of V and V^* with its dual basis. Then by straightforward computations we can show that \mathcal{A}_Θ^0 corresponds to the matrix:

$$A_\Theta := \begin{pmatrix} \zeta_t \zeta_b - 1 & \zeta_t(1 - \zeta_t \zeta_b) & \zeta_b - \zeta_t \\ \zeta_b - \zeta_t & 1 - \zeta_t \zeta_b & \zeta_t \zeta_b - 1 \\ 0 & 1 - \zeta_t^2 & 0 \end{pmatrix}$$

whereas the polarity \mathcal{D}_Θ^0 corresponds to the following symmetric matrix of positive definite:

$$D_\Theta := \begin{pmatrix} 1 & -\zeta_t & -\zeta_b \\ -\zeta_t & 1 & \zeta_t \zeta_b \\ -\zeta_b & \zeta_t \zeta_b & 1 \end{pmatrix}$$

□

Remark 4.6. Therefore, at the level of marked boxes we have:

$$\begin{aligned} [\Theta] &\xrightarrow{\mathcal{A}_\Theta^0} \varrho_1[\Theta] \xrightarrow{\mathcal{A}_\Theta^0} \varrho_1^2[\Theta] \xrightarrow{\mathcal{A}_\Theta^0} [\Theta] \\ [\Theta] &\xrightarrow{\mathcal{D}_\Theta^0} i[\Theta] \xrightarrow{\mathcal{D}_\Theta^0} [\Theta] \end{aligned}$$

Notice that \mathcal{A}_Θ^0 and \mathcal{D}_Θ^0 only depend on the marked box $[\Theta]$.

Theorem 4.7 (Schwartz representation Theorem). *Let $[\Theta]$ be a convex marked box. Put the labels on \mathcal{L}_o as in Section 4.1 so that $[\Theta](e_0) = [\Theta]$. Then there is a faithful representation $\rho_\Theta : \mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathcal{G}$ such that for every Farey geodesic $e \in \mathcal{L}_o$ and every $\gamma \in \mathrm{PSL}(2, \mathbb{Z})$, the following ρ_Θ -equivariant property holds:*

$$[\Theta](\gamma e) = \rho_\Theta(\gamma)([\Theta](e))$$

Proof. Recall (see Section 4.1) that

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle I, R \mid I^2 = 1, R^3 = 1 \rangle.$$

Hence there is a unique representation $\rho_\Theta : \mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathcal{G}$ so that:

$$\rho_\Theta(R) = \mathcal{A}_\Theta^0 \in \mathcal{H} \quad \text{and} \quad \rho_\Theta(I) = \mathcal{D}_\Theta^0 \in \mathcal{G} \setminus \mathcal{H}$$

where the transformations \mathcal{A}_Θ^0 and \mathcal{D}_Θ^0 is defined in Lemma 4.5. Once observed the identities $Re_0 = \varrho_1 e_0$ and $Ie_0 = ie_0$, the ρ_Θ -equivariant property is obviously satisfied for $e = e_0$ and $\gamma = R$ or I . Let now e be any other Farey geodesic. Then:

$$\begin{aligned} [\Theta](Re) &= [\Theta](R(\sigma e_0)) \text{ for some } \sigma \text{ in } \mathfrak{S} \\ &= [\Theta](\sigma(Re_0)) \text{ (the actions of } \mathfrak{S} \text{ and } \mathrm{PSL}(2, \mathbb{Z}) \text{ on } \mathcal{L}_o \text{ commute)} \\ &= \sigma[\Theta](Re_0) \text{ (by the construction of the labeling on } \mathcal{L}_o) \\ &= \sigma \mathcal{A}_\Theta^0([\Theta](e_0)) \text{ (the } \rho_\Theta\text{-equivariant property holds for } \gamma = R, e = e_0) \\ &= \mathcal{A}_\Theta^0(\sigma[\Theta](e_0)) \text{ (the actions of } \mathfrak{S} \text{ and } \mathcal{H} \text{ on } \mathcal{CM} \text{ commute)} \\ &= \rho_\Theta(R)([\Theta](e)) \end{aligned}$$

Hence, the ρ_Θ -equivariant property holds for $\gamma = R$ and for every $e \in \mathcal{L}_o$. We can check this property for I in a similar way, applying the fact that the actions of \mathcal{G} and \mathfrak{S} on marked boxes

commute each other for the last step (whereas for R we only need the fact that \mathfrak{G} commutes with projective transformations).

Now, the general case follows from the facts that R and I generates $\mathrm{PSL}(2, \mathbb{Z})$: If γ is an element of $\mathrm{PSL}(2, \mathbb{Z})$ for which we have $[\Theta](\gamma e) = \rho_\Theta(\gamma)([\Theta](e))$ for every $e \in \mathcal{L}_o$, then:

$$\begin{aligned} [\Theta](\gamma I e) &= \rho_\Theta(\gamma)([\Theta](I e)) \\ &= \rho_\Theta(\gamma)(\rho_\Theta(I)([\Theta](e))) \\ &= \rho_\Theta(\gamma I)([\Theta](e)) \end{aligned}$$

and similarly $[\Theta](\gamma R e) = \rho_\Theta(\gamma R)([\Theta](e))$, completing the proof by induction on the word length of γ in the letters R and I . \square

We call the representation $\rho_\Theta : \mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathcal{G}$ the *Schwartz representation*.

4.3. The Schwartz map. Recall that in Section 4.1, for each convex marked box $[\Theta]$, we attach the labels, which are the elements of the orbit of $[\Theta]$ under \mathfrak{G} , to the Farey geodesics. As we mentioned in Remark 4.4, two Farey geodesics have the same tail point in $\partial\mathbb{H}^2$ if and only if the labels of these geodesics are marked boxes with the same top flag. Therefore, it gives us two ρ_Θ -equivariant maps $\varphi : \mathbb{Q} \cup \{\infty\} \rightarrow \mathbf{P}(V)$ and $\varphi^* : \mathbb{Q} \cup \{\infty\} \rightarrow \mathbf{P}(V^*)$, and moreover the map φ (resp. φ^*) can be extended to an injective ρ_Θ -equivariant continuous map $\varphi_o : \partial\mathbb{H}^2 \rightarrow \mathbf{P}(V)$ (resp. $\varphi_o^* : \partial\mathbb{H}^2 \rightarrow \mathbf{P}(V^*)$) (see Schwartz [13, Theorem 3.2]). The maps φ_o and φ_o^* combine to a ρ_Θ -equivariant map, which we call the *Schwartz map*,

$$\Phi := (\varphi_o, \varphi_o^*) : \partial\mathbb{H}^2 \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*).$$

4.4. The case of special marked boxes. We closely look at the Schwartz representation ρ_Θ and the Schwartz map Φ in the case when Θ is a special marked box. In the Θ -basis of V , the projective transformation \mathcal{A}_Θ^0 corresponds to:

$$A = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & -1 & 0 \end{array} \right)$$

whereas the polarity \mathcal{D}_Θ^0 is expressed by the identity matrix Id_3 . Hence, the image of IRI under ρ_Θ corresponds to the inverse of the transpose of A :

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & -1 \end{array} \right)$$

If we recall that the element R and IRI of $\mathrm{PSL}(2, \mathbb{Z})_o$ is the equivalence class of the matrix:

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

respectively, then we can see that the restriction of ρ_Θ to $\mathrm{PSL}(2, \mathbb{Z})_o$ is the usual linear action of $\mathrm{PSL}(2, \mathbb{Z})_o$ on the affine plane $\{[x : y : z] \in \mathbf{P}(V) \mid x \neq 0\}$. It is true only on $\mathrm{PSL}(2, \mathbb{Z})_o$, not on $\mathrm{PSL}(2, \mathbb{Z})$: the image of I is the polarity associated to the inner product on V for which the Θ -basis of V is orthogonal.

In the case when $[\Theta]$ is special, the map φ_o defined in Section 4.3 is the canonical identification between $\partial\mathbb{H}^2$ and the line $L := \{x = 0\}$ in $\mathbf{P}(V)$, and the image of the Schwartz map Φ is the set of flags $([v], [v^*])$ such that $[v] \in L$ and $[v^*]$ is the line though the points $[v]$ and $[1 : 0 : 0]$.

4.5. Opening the cusps. In the previous subsections, the role of the Farey geodesics is purely combinatorial, except for the definition of the Schwartz map. We can replace the Farey lamination \mathcal{L}_o , which is the set of Farey geodesics, by any other geodesic lamination \mathcal{L} obtained by “opening the cusps” (see Figure 9). The ideal triangles become hyperideal triangles, which means that these triangles are bounded by three geodesics in \mathbb{H}^2 , but now these geodesics have no common point in $\partial\mathbb{H}^2$. The lamination \mathcal{L} is still preserved by a discrete subgroup Γ of $\text{Isom}(\mathbb{H}^2)$, which is isomorphic to $\text{PSL}(2, \mathbb{Z})$ but which is now convex cocompact.

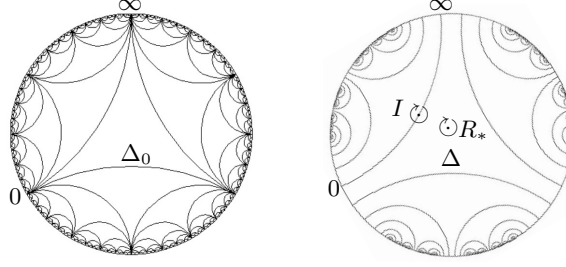


FIGURE 9. The Farey lamination \mathcal{L}_o and the new lamination \mathcal{L} obtained by opening the cusps.

One way to operate this modification is to pick up a hyperideal triangle Δ containing Δ_0 such that Δ still admits the side $e_0 = [\infty, 0]$ but the other two sides are pushed away on the right. The discrete group Γ is then generated by I and the unique (clockwise) rotation R_* of order 3 preserving Δ . Here, we just have to adjust Δ so that the projection of the “center” of the rotation R_* on $e_0 = [\infty, 0]$ is the fixed point of I .

All the discussions in the previous subsections remain true if we interpret the notion of “rotating around the head or tail point” in the appropriate (and obvious) way. In particular, in the quotient surface $\Gamma \backslash \mathbb{H}^2$, the leaves of \mathcal{L} project to wandering geodesics connecting two hyperbolic ends, and for two leaves e, e' of \mathcal{L} , the labels $[\Theta](e)$ and $[\Theta](e')$ have the same bottom if and only if e and e' have tails in the same connected component of $\partial\mathbb{H}^2 \setminus \Lambda_\Gamma$, where Λ_Γ is the limit set of Γ . As a consequence, we still have:

Theorem 4.8 (Modified Schwartz representation Theorem). *Let $[\Theta]$ be a convex marked box. Label the oriented leaves of \mathcal{L} , in a way similar to the labeling of \mathcal{L}_o defined in Section 4.1, so that $[\Theta](e_0) = [\Theta]$. Then there is a faithful representation $\rho_\Theta : \Gamma \rightarrow \mathcal{G}$ such that for every leaf $e \in \mathcal{L}$ and every $\gamma \in \Gamma$ we have:*

$$[\Theta](\gamma e) = \rho_\Theta(\gamma)([\Theta](e))$$

This modified representation is the one obtained by the original Schwartz representation composed with the obvious isomorphism between Γ and $\text{PSL}(2, \mathbb{Z})$, and therefore the original and the modified representations are essentially the same. The main difference is that now the ρ_Θ -equivariant map, called the *modified Schwartz map*,

$$\Phi := (\varphi_o, \varphi_o^*) : \Lambda_\Gamma \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$$

obtained by composing the original Schwartz map with the collapsing map $\Lambda_\Gamma \rightarrow \partial\mathbb{H}^2$ is not injective: It has the same value on the two extremities of each connected component of $\partial\mathbb{H}^2 \setminus \Lambda_\Gamma$.

5. ANOSOV REPRESENTATIONS

The theory of Anosov representations was introduced by Labourie [9] in order to study representations of closed surface groups, and later it was studied by Guichard and Wienhard [6] for finitely generated Gromov-hyperbolic groups. The definition of Anosov representation involves a pair of equivariant maps from the Gromov boundary of the group into certain compact homogeneous spaces (compare with Barbot [1]).

5.1. Definition and properties of Anosov representations. Given $x \in \mathbf{P}(V)$, let $Q_x(V)$ be the space of norms on the tangent space $T_x \mathbf{P}(V)$ at x . Similarly, given $X \in \mathbf{P}(V^*)$, let $Q_X(V^*)$ be the space of norms on the tangent space $T_X \mathbf{P}(V^*)$ at X . Here, a norm is Finsler not necessarily Riemannian. We denote by $Q(V)$ the bundle of base $\mathbf{P}(V)$ with fiber $Q_x(V)$ over $x \in \mathbf{P}(V)$, and by $Q(V^*)$ the bundle of base $\mathbf{P}(V^*)$ with fiber $Q_X(V^*)$ over $X \in \mathbf{P}(V^*)$.

For each convex cocompact subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$, we denote by Λ_Γ the limit set of Γ and by $\Omega(\phi^t)$ the nonwandering set of the geodesic flow ϕ^t on the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^2)$ of $\Gamma \backslash \mathbb{H}^2$: It is the projection of the union in $T^1(\mathbb{H}^2)$ of the orbits of the geodesic flow corresponding to geodesics with tail and head in Λ_Γ .

Definition 5.1. Let Γ be a convex cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A homomorphism $\rho : \Gamma \rightarrow \mathrm{PGL}(V)$ is a $(\mathrm{PGL}(V), \mathbf{P}(V))$ -Anosov representation if there are

- (i) a Γ -equivariant map $\Phi = (\varphi, \varphi^*) : \Lambda_\Gamma \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$, and
- (ii) two maps $\nu_+ : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V)$ and $\nu_- : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V^*)$ such that for every nonwandering geodesic $c : \mathbb{R} \rightarrow \mathbb{H}^2$ joining two points $c_-, c_+ \in \Lambda_\Gamma$, the following exponential increasing/decreasing property holds:
 - for every $v \in T_{\varphi(c_+)} \mathbf{P}(V)$, the size of v for the norm $\nu_+(c(t), c'(t))$ increases exponentially with t ,
 - for every $v \in T_{\varphi^*(c_-)} \mathbf{P}(V^*)$, the size of v for the norm $\nu_-(c(t), c'(t))$ decreases exponentially with t .

Remark 5.2. Technically, the norms ν_\pm in item (ii) do not need to depend continuously on $(x, v) \in \Omega(\phi^t)$. The continuity, in fact, follows from the exponential increasing/decreasing property. It might be difficult to directly check this property, but there is a simpler criteria: it suffices to prove that there is a time $T > 0$ such that at every time t :

$$\begin{aligned} \nu_+(c(t+T), c'(t+T)) &> 2\nu_+(c(t), c'(t)) \\ \nu_-(c(t+T), c'(t+T)) &< \frac{1}{2}\nu_-(c(t), c'(t)) \end{aligned}$$

For a proof of this folklore, see e.g. Barbot–Mérigot [2, Proposition 5.5].

Since the group Γ of this definition is a Gromov-hyperbolic group realized as a convex cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$, its Gromov boundary $\partial\Gamma$ is Γ -equivariantly homeomorphic to its limit set Λ_Γ . The reader can find more information about Gromov-hyperbolic groups in Ghys–de la Harpe [4], Gromov [5] and Kapovich–Benakli [7].

We denote by $\mathrm{Rep}(\Gamma, \mathrm{PGL}(V))$ the space of (conjugacy classes of) representations of Γ into $\mathrm{PGL}(V)$, and by $\mathrm{Rep}_A(\Gamma, \mathrm{PGL}(V))$ the space of (conjugacy classes of) Anosov representations in $\mathrm{Rep}(\Gamma, \mathrm{PGL}(V))$. Here are some basic properties of Anosov representations (see e.g. Barbot [1], Guichard–Wienhard [6] or Labourie [9]).

- (1) $\mathrm{Rep}_A(\Gamma, \mathrm{PGL}(V))$ is an open set in $\mathrm{Rep}(\Gamma, \mathrm{PGL}(V))$.
- (2) Every Anosov representation is discrete and faithful.
- (3) The maps φ and φ^* are injective.

- (4) If an Anosov representation is irreducible (i.e. it does not preserve a non-trivial linear subspace of V), then φ (resp. φ^*) is the unique Γ -equivariant map from $\partial\Gamma$ into $\mathbf{P}(V)$ (resp. $\mathbf{P}(V^*)$).

Remark 5.3. There is also a dual concept: $(\mathrm{PGL}(V), \mathbf{P}(V^*))$ -Anosov representation when the role of $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ are switched. It corresponds also to reverse the orientation of the geodesic flow. In this paper, by Anosov representation, we mean a $(\mathrm{PGL}(V), \mathbf{P}(V))$ -Anosov representation.

5.2. Schwartz representations are not Anosov. Let $\rho_\Theta : \Gamma \rightarrow \mathcal{G}$ be the (modified) Schwartz representation associated to a convex marked box $[\Theta]$, and let

$$\Phi = (\varphi_o, \varphi_o^*) : \partial\Gamma \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$$

be the Γ -equivariant (modified) Schwartz map defined in Section 4.5 (remember also Section 4.3) which seems to indicate that ρ_Θ might be Anosov. But it cannot be because of a simple reason: The target group of ρ_Θ is the group \mathcal{G} of projective symmetries, not merely the group \mathcal{H} of projective transformations, which can be identified with $\mathrm{PGL}(V)$. Hence, some elements of Γ would exchange the stable and unstable foliations, which is absolutely nonsense. Nevertheless, this inconvenience can be easily eluded, simply by restricting of ρ_Θ to the unique index 2 subgroup Γ_o of Γ (see Remark 4.1).

More seriously, there is another fact making clear that ρ_Θ is not Anosov: if $[\Theta]$ is not special, then ρ_Θ is irreducible and the Γ -equivariant map from $\partial\Gamma$ into \mathcal{F} must be the map Φ . However, this map is not injective, whereas according to the item (4) in Section 5.1, it should be if ρ_Θ is Anosov.

6. A NEW FAMILY OF REPRESENTATIONS OF $\mathrm{PSL}(2, \mathbb{Z})_o$

In order to show that Schwartz representations are limits of Anosov representations, we build paths (families) of Anosov representations that end in Schwartz representations. With this goal in mind, we first introduce a new group of transformations of marked boxes and consequently we obtain an analog of Theorem 4.8 (Schwartz representation Theorem).

6.1. A new group of transformations of marked boxes. Let

$$\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$$

be a overmarked box. For each pair (ε, δ) of real numbers, let $\sigma_{(\varepsilon, \delta)} : \mathcal{CSM} \rightarrow \mathcal{CSM}$ be a new transformation of overmarked boxes such that the image of Θ is given by applying the matrix

$$\Sigma_{(\varepsilon, \delta)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\delta} \cosh(\varepsilon) & -\sinh(\varepsilon) \\ 0 & -\sinh(\varepsilon) & e^{\delta} \cosh(\varepsilon) \end{pmatrix}$$

(for each Θ -basis of V) to Θ (see Figure 10).

Lemma 6.1. The transformation $\sigma_{(\varepsilon, \delta)}$ commutes with j and therefore it acts on \mathcal{CM} .

Proof. The projective transformation J given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

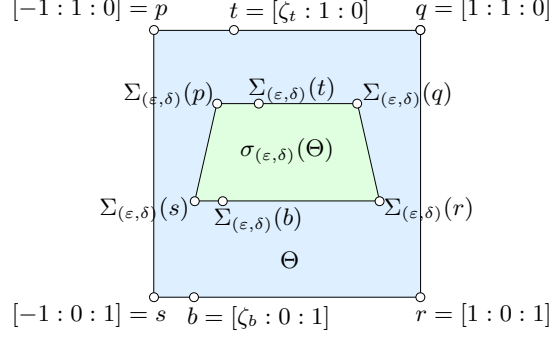


FIGURE 10. New permutation $\sigma_{(\varepsilon, \delta)}$ and a convex interior of $\sigma_{(\varepsilon, \delta)}(\Theta)$ in $\mathbf{P}(V)$ is drawn in green when Θ is convex.

for each Θ -basis of V sends the points p, q, r, s onto q, p, s, r (in this order). It is obvious that J is an involution and $J\Sigma_{(\varepsilon, \delta)}J^{-1} = \Sigma_{(\varepsilon, \delta)}$.

$$\begin{aligned}
 (j \circ \sigma_{(\varepsilon, \delta)} \circ j)(\Theta) &= (j \circ \sigma_{(\varepsilon, \delta)} \circ j)((p, q, r, s; t, b), (P, Q, R, S; T, B)) \\
 &= (j \circ \sigma_{(\varepsilon, \delta)})((q, p, s, r; t, b), (Q, P, S, R; T, B)) \\
 &= j((\check{q}, \check{p}, \check{s}, \check{r}; \check{t}, \check{b}), (\check{Q}, \check{P}, \check{S}, \check{R}; \check{T}, \check{B})) \\
 &= ((\check{p}, \check{q}, \check{r}, \check{s}; \check{t}, \check{b}), (\check{P}, \check{Q}, \check{R}, \check{S}; \check{T}, \check{B})) \\
 &= \sigma_{(\varepsilon, \delta)}(\Theta),
 \end{aligned}$$

where $\check{x} = (J\Sigma_{(\varepsilon, \delta)}J^{-1})(x)$ for $x \in \mathbf{P}(V)$ and $\check{X} = (J\Sigma_{(\varepsilon, \delta)}J^{-1})^*(X)$ for $X \in \mathbf{P}(V^*)$. \square

Remark 6.2. It is easy to see that $\sigma_{(\varepsilon, \delta)}$ commutes with elements of \mathcal{H} (projective transformations), however *it does not commute* with elements of $\mathcal{G} \setminus \mathcal{H}$ (dualities) acting on \mathcal{CM} .

Recall that the transformation i is the involution on \mathcal{CM} defined in Section 3.4.

Lemma 6.3. The following relations hold:

$$\sigma_{(-\varepsilon, -\delta)} = \sigma_{(\varepsilon, \delta)}^{-1} \quad \text{and} \quad i\sigma_{(\varepsilon, \delta)} = \sigma_{(-\varepsilon, -\delta)}i$$

Proof. The first relation easily follows from the fact that $\Sigma_{(-\varepsilon, -\delta)} = \Sigma_{(\varepsilon, \delta)}^{-1}$. A proof of the second relation is similar to the proof of Lemma 6.1: The projective transformation K given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

for each Θ -basis of V sends the points p, q, r, s onto s, p, q, r , respectively. An easy computation shows that $K\Sigma_{(\varepsilon, \delta)}K^{-1} = \Sigma_{(-\varepsilon, -\delta)}$, and therefore

$$\begin{aligned}
 (j \circ i \circ \sigma_{(\varepsilon, \delta)} \circ i)(\Theta) &= (j \circ i \circ \sigma_{(\varepsilon, \delta)} \circ i)((p, q, r, s; t, b), (P, Q, R, S; T, B)) \\
 &= (j \circ i \circ \sigma_{(\varepsilon, \delta)})((s, r, p, q; b, t), (R, S, Q, P; B, T)) \\
 &= (j \circ i)((\check{s}, \check{r}, \check{p}, \check{q}; \check{b}, \check{t}), (\check{R}, \check{S}, \check{Q}, \check{P}; \check{B}, \check{T})) \\
 &= j((\check{q}, \check{p}, \check{s}, \check{r}; \check{t}, \check{b}), (\check{Q}, \check{P}, \check{S}, \check{R}; \check{T}, \check{B})) \\
 &= ((\check{p}, \check{q}, \check{r}, \check{s}; \check{t}, \check{b}), (\check{P}, \check{Q}, \check{R}, \check{S}; \check{T}, \check{B})) \\
 &= \sigma_{(-\varepsilon, -\delta)}(\Theta)
 \end{aligned}$$

where $\check{x} = (K\Sigma_{(\varepsilon,\delta)}K^{-1})(x)$ for $x \in \mathbf{P}(V)$ and $\check{X} = (K\Sigma_{(\varepsilon,\delta)}K^{-1})^*(X)$ for $X \in \mathbf{P}(V^*)$. \square

A simple observation is that with respect to the Θ -basis of V , the point $[x : y : z] \in \mathbf{P}(V)$ is in the convex interior of a convex marked box $[\Theta]$ if and only if

$$(3) \quad y + z \neq 0, \quad \left| \frac{x}{y+z} \right| < 1 \quad \text{and} \quad \left| \frac{y-z}{y+z} \right| < 1.$$

Define the function $f(\varepsilon, \delta) = e^{-\delta} \cosh(\varepsilon) - \sinh(\varepsilon) - 1$ and the region

$$(4) \quad \mathcal{R} = \{(\varepsilon, \delta) \in \mathbb{R}^2 \mid f(\varepsilon, \delta) \geq 0 \text{ and } f(\varepsilon, -\delta) \geq 0\}$$

of \mathbb{R}^2 (See Figure 11). Then it follows by Equation (3) that for each $(\varepsilon, \delta) \in \mathbb{R}^2$, the convex interior of $\sigma_{\varepsilon,\delta}(\Theta)$ is contained in the convex interior of Θ if and only if $(\varepsilon, \delta) \in \mathcal{R}$.

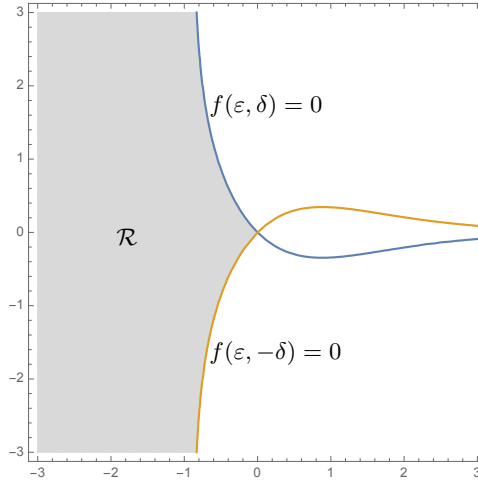


FIGURE 11. The region \mathcal{R} is drawn in grey.

From now on, for the simplicity of the notation, let $\lambda = (\varepsilon, \delta)$. For example, $\sigma_\lambda = \sigma_{(\varepsilon,\delta)}$. Let us introduce three more new transformations on \mathcal{CM} as follows:

$$i^\lambda := \sigma_\lambda i, \quad \tau_1^\lambda := \sigma_\lambda \tau_1, \quad \tau_2^\lambda := \sigma_\lambda \tau_2.$$

Lemma 6.4. The following relations hold:

$$(i^\lambda)^2 = 1, \quad \tau_1^\lambda i^\lambda \tau_2^\lambda = i^\lambda, \quad \tau_2^\lambda i^\lambda \tau_1^\lambda = i^\lambda, \quad \tau_1^\lambda i^\lambda \tau_1^\lambda = \tau_2^\lambda, \quad \tau_2^\lambda i^\lambda \tau_2^\lambda = \tau_1^\lambda, \quad (i^\lambda \tau_1^\lambda)^3 = 1.$$

Proof. The proof follows directly from Lemma 3.6 and the relation $i\sigma_\lambda = \sigma_\lambda^{-1}i$. \square

Thus, by Lemma 6.4, the inverses of i^λ , τ_1^λ and τ_2^λ are

$$(i^\lambda)^{-1} = i^\lambda, \quad (\tau_1^\lambda)^{-1} = i^\lambda \tau_2^\lambda i^\lambda, \quad (\tau_2^\lambda)^{-1} = i^\lambda \tau_1^\lambda i^\lambda.$$

As a result, the semigroup \mathfrak{G}^λ of $\mathcal{S}(\mathcal{CM})$ generated by i^λ , τ_1^λ and τ_2^λ is in fact a group. The key point is that if $\lambda \in \mathcal{R}$, then for every convex marked box $[\Theta]$, we still have $[\tau_1^\lambda(\Theta)]^\circ \subsetneq [\Theta]^\circ$, $[\tau_2^\lambda(\Theta)]^\circ \subsetneq [\Theta]^\circ$ and $[i^\lambda(\Theta)]^\circ \cap [\Theta]^\circ = \emptyset$ and furthermore if $\lambda \in \mathcal{R}^\circ$, the interior of \mathcal{R} , then we have the same properties but now for the *closures* of the interiors of the marked boxes. The Anosov character of new representations we build is a consequence of this stronger property.

Anyway, by the same arguments as in the case when $\lambda = (0, 0)$, we can easily deduce:

Lemma 6.5. If $\lambda \in \mathcal{R}$, the action of \mathfrak{G}^λ on convex marked boxes is free. \square

Hence if $\lambda \in \mathcal{R}$, then we have the group presentation:

$$\mathfrak{G}^\lambda = \langle i^\lambda, \tau_1^\lambda \mid (i^\lambda)^2 = 1, (i^\lambda \tau_1^\lambda)^3 = 1 \rangle$$

and thus \mathfrak{G}^λ is isomorphic to the modular group. An important remark is

$$i^\lambda \tau_1^\lambda = \sigma_\lambda i \sigma_\lambda \tau_1 = i \tau_1,$$

and so we should rewrite the presentation in the following form:

$$\mathfrak{G}^\lambda = \langle i^\lambda, \varrho_1 \mid (i^\lambda)^2 = 1, \varrho_1^3 = 1 \rangle$$

where $\varrho_1 = i \tau_1$ is a Schwartz transformation of marked boxes defined in Section 3.4. In other words, \mathfrak{G}^λ is simply obtained from \mathfrak{G} by replacing i by i^λ , and keeping ϱ_1 the same.

Remark 6.6. If $\lambda \notin \mathcal{R}$, then the situation is completely different. In this case, it is not clear that \mathfrak{G}^λ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$, and that its action on \mathcal{CM} is free. However, it is not important, and in the sequel, when $\lambda \notin \mathcal{R}$, by \mathfrak{G}^λ we mean the group $\mathbb{Z}_2 * \mathbb{Z}_3$ but acting (maybe not freely) on the set of marked boxes. Anyway, we are mostly interested in the case when $\lambda \in \mathcal{R}^\circ$ because it corresponds to an Anosov representation.

6.2. New representations. Given a convex marked box $[\Theta]$ and $\lambda = (\varepsilon, \delta) \in \mathbb{R}^2$, let us look at the convex cocompact subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ and the lamination \mathcal{L} of \mathbb{H}^2 introduced in Section 4.5, and the new group \mathfrak{G}^λ of transformations of \mathcal{CM} .

We cannot directly prove an analog of Theorem 4.7 since it is not true anymore that new transformations of marked boxes commute with dualities. In order to avoid this inconvenience, we have to restrict the domain of new representations to the subgroup Γ_o of Γ :

$$\Gamma_o = \langle R_*, IR_* I \mid R_*^3 = 1, (IR_* I)^3 = 1 \rangle.$$

This subgroup Γ_o is isomorphic to $\mathbb{Z}_3 * \mathbb{Z}_3$, it has index 2 in Γ , and it is the image of $\mathrm{PSL}(2, \mathbb{Z})_o$ under the isomorphism between $\mathrm{PSL}(2, \mathbb{Z})$ and Γ . It preserves \mathcal{L} but its action on oriented leaves of \mathcal{L} is not transitive. However, the action of Γ_o on *non-oriented* leaves of \mathcal{L} is simply transitive. Now, the index 2 subgroup \mathfrak{G}_o^λ generated by $\varrho_1 = i^\lambda \tau_1^\lambda$ and $\tau_1^\lambda i^\lambda$ also acts on the set of non-oriented leaves of \mathcal{L} . Moreover, the action of \mathfrak{G}_o^λ is simply transitive and still commutes with the action of Γ_o (see Remark 4.3).

In order to define our new representation of Γ_o (*not* Γ), we only need:

Lemma 6.7. Let Θ be a convex overmarked box. Then

1. there is only one projective transformation $\mathcal{A}_\Theta^\lambda \in \mathcal{H}$ such that:

$$\Theta \xrightarrow{\mathcal{A}_\Theta^\lambda} j i^\lambda \tau_1^\lambda \Theta \xrightarrow{\mathcal{A}_\Theta^\lambda} (i^\lambda \tau_1^\lambda)^2 \Theta \xrightarrow{\mathcal{A}_\Theta^\lambda} j \Theta$$

2. there is only one projective transformation $\mathcal{B}_\Theta^\lambda \in \mathcal{H}$ such that:

$$\Theta \xrightarrow{\mathcal{B}_\Theta^\lambda} j \tau_1^\lambda i^\lambda \Theta \xrightarrow{\mathcal{B}_\Theta^\lambda} (\tau_1^\lambda i^\lambda)^2 \Theta \xrightarrow{\mathcal{B}_\Theta^\lambda} j \Theta$$

Proof. The first item is exactly the first item of Lemma 4.5 since $i^\lambda \tau_1^\lambda = i \tau_1 = \varrho_1$, hence $\mathcal{A}_\Theta^\lambda$ is precisely \mathcal{A}_Θ^0 .

The second item is a corollary of the first item: apply the first item to $i^\lambda \Theta$, and use the fact that i^λ commutes with $\mathcal{A}_\Theta^\lambda$. However, we give an alternative proof, which is useful for the later discussion: If we recall that Σ_λ is the projective transformation of $\mathbf{P}(V)$ defined in Section 6.1 and \mathcal{B}_Θ^0 is the image of $IR_* I$ under ρ_Θ in Theorem 4.8, then the projective transformation $\mathcal{B}_\Theta^\lambda$ is actually $\Sigma_\lambda^{-1} \mathcal{B}_\Theta^0 \Sigma_\lambda$.

Let $\varrho'_1 = \tau_1 i$ and look at the following diagram, which arises from the fact that Σ_λ^{-1} commutes with every elementary transformation of marked boxes:

$$\begin{array}{ccccccc}
\Theta & \xrightarrow{\mathcal{B}_\Theta^0} & j\varrho'_1\Theta & \xrightarrow{\mathcal{B}_\Theta^0} & (\varrho'_1)^2\Theta & \xrightarrow{\mathcal{B}_\Theta^0} & j\Theta \\
\Sigma_\lambda^{-1} \downarrow & & \Sigma_\lambda^{-1} \downarrow & & \Sigma_\lambda^{-1} \downarrow & & \Sigma_\lambda^{-1} \downarrow \\
\sigma_\lambda^{-1}\Theta & & j\varrho'_1\sigma_\lambda^{-1}\Theta & & (\varrho'_1)^2\sigma_\lambda^{-1}\Theta & & j\sigma_\lambda^{-1}\Theta
\end{array}$$

Therefore:

$$\sigma_\lambda^{-1}\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} j\varrho'_1\sigma_\lambda^{-1}\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} (\varrho'_1)^2\sigma_\lambda^{-1}\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} j\sigma_\lambda^{-1}\Theta$$

Since σ_λ and the projective transformation $\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda$ commute each other:

$$\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} j\sigma_\lambda\varrho'_1\sigma_\lambda^{-1}\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} \sigma_\lambda(\varrho'_1)^2\sigma_\lambda^{-1}\Theta \xrightarrow{\Sigma_\lambda^{-1}\mathcal{B}_\Theta^0\Sigma_\lambda} j\Theta$$

Our claim then follows since:

$$\begin{aligned}
\sigma_\lambda\varrho'_1\sigma_\lambda^{-1} &= \sigma_\lambda\tau_1 i\sigma_\lambda^{-1} \\
&= \tau_1^\lambda(\sigma_\lambda i) \\
&= \tau_1^\lambda i^\lambda
\end{aligned}$$

□

The next Theorem is similar to Theorem 4.7 (better to say, Theorem 4.8), but now the leaves of \mathcal{L} must be understood as *non-oriented* geodesics.

Theorem 6.8. *Let $[\Theta]$ be a convex marked box and let $\lambda \in \mathbb{R}^2$. Then there is a representation $\rho_\Theta^\lambda : \Gamma_o \rightarrow \mathcal{H} \subset \mathcal{G}$ such that for every (non-oriented) leaf e of \mathcal{L} and every $\gamma \in \Gamma$ we have:*

$$[\Theta](\gamma e) = \rho_\Theta^\lambda(\gamma)([\Theta](e))$$

Moreover, if $\lambda \in \mathcal{R}$, then ρ_Θ^λ is faithful.

Proof. Define $\rho_\Theta^\lambda : \Gamma_o \rightarrow \mathcal{H} \subset \mathcal{G}$ by requiring:

$$\rho_\Theta^\lambda(R_*) = \mathcal{A}_\Theta^\lambda \in \mathcal{H} \quad \text{and} \quad \rho_\Theta^\lambda(IR_*I) = \mathcal{B}_\Theta^\lambda \in \mathcal{H}$$

where $\mathcal{A}_\Theta^\lambda$ and $\mathcal{B}_\Theta^\lambda$ are the projective transformations defined in Lemma 6.7. Here we can apply the arguments in the proof of Theorem 4.7 since no dualities are involved - in that proof, we emphasized that the commutativity between the actions of \mathfrak{G} and dualities was used only for defining the image of the involution I . □

7. A SPECIAL NORM ASSOCIATED TO MARKED BOXES

In this section, we will show that given a convex marked box $[\Theta]$, we can define a special norm associated to $[\Theta]$. For this purpose, we use the Hilbert metric on properly convex domains. The reader can find more information about the Hilbert metric in Marquis [10] or Orenstein [12].

Let D be a *properly convex* domain in $\mathbf{P}(V)$, i.e. there is an affine chart \mathbb{A} of $\mathbf{P}(V)$ such that the closure \overline{D} of D is contained in \mathbb{A} and D is convex in \mathbb{A} in the usual sense. For distinct points $x, y \in D$, let p and q be the intersection points of the line xy with the boundary ∂D in

such a way that a and y separate x and b on the line xy (see Figure 12). The *Hilbert metric* $d_D^h : D \times D \rightarrow [0, +\infty)$ is defined by:

$$d_D^h(x, y) = \frac{1}{2} \log([a, x; y, b]) \quad \text{for every } x, y \in D$$

where $[a, x; y, b]$ is the cross-ratio of the four points $a, x, y, b \in \mathbf{P}(V)$.

The Hilbert metric can be also defined by a Finsler norm on the tangent space $T_x D$ at each point $x \in D$: Let $x \in D$, $v \in T_x D$ and let p^+ (resp. p^-) be the intersection point of ∂D with the half-line determined by x and v (resp. $-v$) (see Figure 12).

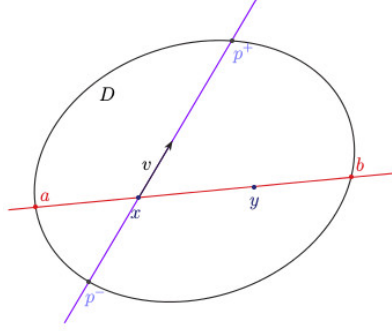


FIGURE 12. The Hilbert metric

The *Hilbert norm* of v , denoted by $\|v\|_D^h$, is the Finsler norm defined by:

$$\|v\|_D^h = \frac{|v|}{2} \left(\frac{1}{|x - p^-|} + \frac{1}{|x - p^+|} \right)$$

where $|\cdot|$ is any Euclidean norm on any affine chart \mathbb{A} containing \overline{D} . The following lemma demonstrates the expansion property of the Hilbert metric by inclusion:

Lemma 7.1. Let D_1 and D_2 be properly convex domains in $\mathbf{P}(V)$. Assume that $\overline{D_2} \subset D_1$. Then there is a constant $C > 1$ such that

- (1) $d_{D_2}^h(x, y) \geq C d_{D_1}^h(x, y)$ for every $x, y \in D_2$,
- (2) $\|v\|_{D_2}^h \geq C \|v\|_{D_1}^h$ for every $x \in D_2$ and for every $v \in T_x D_2 = T_x D_1$.

Proof. See the proof in Orenstein [12, Teorema 7]. □

We define the *distortion* from D_1 to D_2 , denoted by $C(D_2, D_1)$, to be the upper bound of the set of C 's for which the inequalities (1) and (2) in Lemma 7.1 hold.

The following lemma is obvious since projective transformations preserve the cross-ratio.

Lemma 7.2. Let D_1 and D_2 be two properly convex domains in $\mathbf{P}(V)$ such that $\overline{D_2} \subset D_1$, and let g be a projective transformation of $\mathbf{P}(V)$. Then $C(gD_2, gD_1) = C(D_2, D_1)$. □

Moreover:

Lemma 7.3. Let D_1 , D_2 and D_3 be properly convex domains in $\mathbf{P}(V)$ such that $\overline{D_2} \subset D_1$ and $\overline{D_3} \subset D_2$. Then $C(D_3, D_1) \geq C(D_3, D_2) C(D_2, D_1)$. □

Remark 7.4. For each convex marked box $[\Theta]$, the convex interior of $[\Theta]$ (resp. $[\Theta^*]^\circ$) is a properly convex domain in $\mathbf{P}(V)$ (resp. $\mathbf{P}(V^*)$). Hence we can define the Hilbert metric (norm) on $[\Theta]^\circ$ (resp. $[\Theta^*]^\circ$).

8. A FAMILY OF ANOSOV REPRESENTATIONS

In this section, we give the proof of Theorem 1.1. Recall that \mathcal{H} can be identified with $\mathrm{PGL}(V)$ and in Theorem 6.8 we define the representations $\rho_\Theta^\lambda : \Gamma_o \rightarrow \mathcal{H}$. Since the groups $\mathrm{PSL}(2, \mathbb{Z})_o$ and Γ_o are isomorphic, we just have to show that the representations ρ_Θ^λ are Anosov when $\lambda \in \mathcal{R}^\circ$.

From now on, assume that $\lambda \in \mathcal{R}^\circ$. We only need to verify that there are

- (1) a Γ_o -equivariant map $\Phi^\lambda = (\varphi_\lambda, \varphi_\lambda^*) : \Lambda_{\Gamma_o} \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$, and
- (2) two maps $\nu_+ : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V)$ and $\nu_- : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V^*)$ that “carry” the Anosov property of expansion and contraction.

8.1. The equivariant map of new representations. In order to defined the map Φ^λ , we construct $\varphi_\lambda : \Lambda_{\Gamma_o} \rightarrow \mathbf{P}(V)$ and $\varphi_\lambda^* : \Lambda_{\Gamma_o} \rightarrow \mathbf{P}(V^*)$ separately.

We first label the oriented leaves of \mathcal{L} by elements of the orbit of $[\Theta]$ under \mathfrak{G}^λ as in Section 4.1. Let $\alpha \in \Lambda_{\Gamma_o}$ and let c be a Γ -nonwandering oriented geodesic on $T^1(\Gamma \backslash \mathbb{H}^2)$, whose head is α . Since c is nonwandering, it meets infinitely many leaves of \mathcal{L} . We orient each of these geodesics so that c crosses each of them from the right to the left, and denote them by ℓ_m with $m \in \mathbb{Z}$ (see Figure 13). According to Remark 4.4, the labels $[\Theta](\ell_m)$ of these oriented leaves

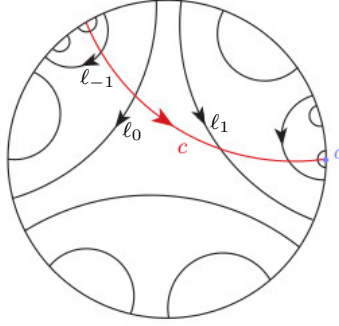


FIGURE 13. A sequence (ℓ_m) for φ_λ

ℓ_m of \mathcal{L} give us a sequence of convex marked boxes $[\Theta_m] := [\Theta](\ell_m)$ satisfying the following nesting property in $\mathbf{P}(V)$:

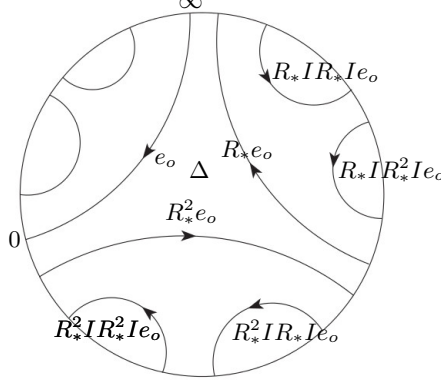
$$(5) \quad \dots \supset [\Theta_{-1}]^\circ \supset [\Theta_0]^\circ \supset [\Theta_1]^\circ \supset \dots \supset [\Theta_m]^\circ \supset \dots$$

However, it does not immediately imply that the intersection $\bigcap_{m \in \mathbb{Z}} [\Theta_m]^\circ$ reduces to one point, but we will show that now by proving that each inclusion is uniform in a certain sense.

Recall the objects e_0 and Δ in Section 4.5. We can assume without loss of generality that the geodesic ℓ_0 is the image of e_0 under some element γ_0 of Γ_o . Now, we forget all the other geodesics with odd index. We claim that for every integer n , the oriented geodesic ℓ_{2n} is in the Γ_o -orbit of ℓ_0 . Furthermore, if γ_n is the unique element of Γ_o for which $\ell_{2n} = \gamma_n e_0$, then we have $\gamma_{n+1} = \gamma_n w$, where w is one of the elements of the following subset of Γ_o :

$$W = \{R_* I R_* I, R_* I R_*^2 I, R_*^2 I R_*^2 I, R_*^2 I R_* I\}$$

Indeed, the image of c under γ_n^{-1} crosses e_0 from the right to the left, hence enters in Δ . Then, it exits Δ from one of the two other sides $R_* e_0$ or $R_*^2 e_0$ (see Figure 14). Observe that these crossings are both from the left to the right, hence ℓ_{2n+1} is the image by γ_n of $R_* e_0$ or $R_*^{-1} e_0$ with the reversed orientation, and therefore ℓ_{2n+1} is not in the orbit of e_0 under Γ_o .

FIGURE 14. The four leaves on the left of e_o at the 2nd step

In the first case, the case when the geodesic crosses R_*e_o , it enters in the triangle $R_*IR_*^2\Delta$, and then exit, from the right to the left, through either $R_*IR_*e_o$ or $R_*IR_*^2e_o$. Since the crossing is in the right direction, we obtain that $\gamma_{n+1} = \gamma_n R_*IR_*I$ or $\gamma_{n+1} = \gamma_n R_*IR_*^2I$.

In the second case, we just have to replace R_* by R_*^{-1} , and we then have $\gamma_{n+1} = \gamma_n R_*^2IR_*^2I$ or $\gamma_{n+1} = \gamma_n R_*^2IR_*I$. Our claim is proved.

Define:

$$(6) \quad C = \min \{ C([\Theta](we_o)^\circ, [\Theta](e_o)^\circ) \mid w \in W \}$$

Then it follows from Lemma 7.2 that for every $n \in \mathbb{Z}$, we have:

$$C([\Theta_{2n+2}]^\circ, [\Theta_{2n}]^\circ) \geq C$$

If we look at all the closures of the convex domains $[\Theta_{2n}]^\circ$, then it is a decreasing sequence of compact sets as n goes to infinity, and hence their intersection is not empty. Assume that the intersection contains two different elements a, b . Let x, y be two distinct elements in the segment $]a, b[$. For every integer n , let $d_n^h(x, y)$ be the Hilbert metric between x and y with respect to the domain $[\Theta_{2n}]^\circ$. According to Lemma 7.3, we have:

$$d_n^h(x, y) \geq C^n d_0^h(x, y)$$

On the other hand, for every n , we have:

$$d_n^h(x, y) \leq \frac{1}{2} \log([a : x; y, b])$$

which is a contradiction. Therefore, the intersection $\bigcap_{m \in \mathbb{Z}} [\Theta_m]^\circ = \bigcap_{n \in \mathbb{Z}} [\Theta_{2n}]^\circ$ is reduced to a single point in $\mathbf{P}(V)$, which we denote by $\psi_\lambda(c)$. By the observation that any other nonwandering oriented geodesic c' whose head is α ultimately intersects the same leaves of \mathcal{L} , it follows that $\psi_\lambda(c) = \psi_\lambda(c')$, i.e. ψ_λ does not depend on the geodesic we select, and only depends on α . Consequently, it gives us a map $\varphi_\lambda : \Lambda_{\Gamma_o} \rightarrow \mathbf{P}(V)$.

In summary, $\varphi_\lambda(\alpha)$ is the intersection of the convex interiors of the labels of leaves of \mathcal{L} crossed by any geodesic with head α .

In a similar way, we define the map $\varphi_\lambda^* : \Lambda_{\Gamma_o} \rightarrow \mathbf{P}(V^*)$. By Remark 3.5, the inclusions of the sequence (5) along the oriented nonwandering geodesic c are reversed when viewed in $\mathbf{P}(V^*)$:

$$\dots \subset [\Theta_{-1}^*]^\circ \subset [\Theta_0^*]^\circ \subset [\Theta_1^*]^\circ \subset \dots \subset [\Theta_m^*]^\circ \subset \dots$$

We can show that this nested sequence of convex domains is again uniform with respect to the Hilbert metrics; in particular, the intersection $\bigcap_{m \in \mathbb{Z}} [\Theta_m^*]^\circ$ is reduced to a single point $\psi_\lambda^*(c)$, and two nonwandering geodesics c and c' sharing the same *tail* α leads to the same value of ψ_λ^* . Hence it defines a map $\varphi_\lambda^* : \Lambda_{\Gamma_o} \rightarrow \mathbf{P}(V^*)$.

The maps φ_λ and φ_λ^* are obviously Γ_o -equivariant, but it is not clear from our construction that they combine to a map in the flag variety, i.e. that $\varphi_\lambda(\alpha)$ is a point in the line $\varphi_\lambda^*(\alpha)$ of $\mathbf{P}(V)$. However, a simple trick makes it obvious: Consider the marked box $[i(\Theta_n)]$: it corresponds to exchange the orientation of the leaves crossed by the geodesic c (see Figure 15):

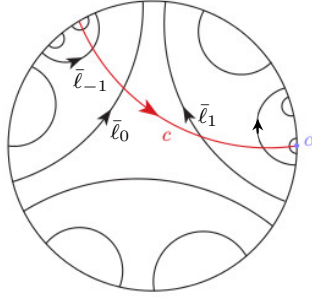


FIGURE 15. A sequence $(\bar{\ell}_m)$ for φ_λ^*

Then we have a nested sequence:

$$\dots \supset [i(\Theta_{-1})^*]^\circ \supset [i(\Theta_0)^*]^\circ \supset [i(\Theta_1)^*]^\circ \supset \dots \supset [i(\Theta_m)^*]^\circ \supset \dots$$

The common intersection point $\bigcap_{m \in \mathbb{Z}} [i(\Theta_m)^*]^\circ$ is clearly $\psi_\lambda^*(\bar{c})$, where \bar{c} is the geodesic c with the reversed orientation. In particular, if α is the head of c , then this intersection point is $\varphi_\lambda^*(\alpha)$.

Now the key point is that the top point t_m^* of each $[i(\Theta_m)^*]$ is the bottom line of $[\Theta_m]$. The bottom points b_m of $[\Theta_m]$ converge to $\psi_\lambda(c)$ whereas t_m^* converge to $\psi_\lambda^*(\bar{c})$. Since every t_m^* contains b_m , the line $\varphi_\lambda^*(\alpha)$ of $\mathbf{P}(V)$ also contains $\varphi_\lambda(\alpha)$. Hence, the maps φ_λ and φ_λ^* combine to a Γ_o -equivariant map:

$$\Phi^\lambda = (\varphi_\lambda, \varphi_\lambda^*) : \Lambda_{\Gamma_o} \rightarrow \mathcal{F} \subset \mathbf{P}(V) \times \mathbf{P}(V^*)$$

Remark 8.1. It is easy to prove that the map $\varphi_\lambda : \Lambda_{\Gamma_o} \subset \partial\mathbb{H}^2 \rightarrow \mathbf{P}(V)$ is continuous: Let $\alpha \in \Lambda_{\Gamma_o}$, and as before, let $[\Theta_m]$ be the labels of the leaf ℓ_m ($m \in \mathbb{Z}$) of \mathcal{L} crossed by a geodesic c with head α . Then for any open neighborhood \mathcal{U} of $\varphi_\lambda(\alpha)$ in $\mathbf{P}(V)$, there exists a marked box $[\Theta_{2n}]$ such that $\varphi_\lambda(\alpha) \in [\Theta_{2n}]^\circ \subset \mathcal{U}$ since $\bigcap_{n \in \mathbb{Z}} [\Theta_{2n}]^\circ$ is a singleton. Hence, if $\beta \in \Lambda_{\Gamma_o}$ is sufficiently close to α , then every geodesic with head β will intersect ℓ_{2n} , and thus $\varphi_\lambda(\beta)$ is contained in the interior of $[\Theta_{2n}]$, which completes the proof.

Similarly, we can show that $\varphi_\lambda^* : \Lambda_{\Gamma_o} \subset \partial\mathbb{H}^2 \rightarrow \mathbf{P}(V^*)$ is continuous.

8.2. The Anosov property of new representations. In this subsection, we construct the maps:

$$\nu_+ : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V) \quad \text{and} \quad \nu_- : \Omega(\phi^t) \subset T^1(\Gamma \backslash \mathbb{H}^2) \rightarrow Q(V^*)$$

The definition is as follows: let $(x, v) \in \Omega(\phi^t)$ and let c be the oriented geodesic such that $c(0) = x$ and $c'(0) = v$. Let c_- (resp. c_+) be the tail (resp. head) of c .

If x lies on a leaf ℓ of \mathcal{L} , which is oriented so that it is crossed by c from the right to the left, then $\varphi_\lambda(c_+)$ (resp. $\varphi_\lambda^*(c_-)$) lies in the convex interior of the label $[\Theta]$ of ℓ (resp. in $[\Theta^*]^\circ$). Define

- $\nu_+(x, v)$ as the Hilbert norm on $T_{\varphi_\lambda(c_+)}\mathbf{P}(V)$ associated to $[\Theta]^\circ$ in $\mathbf{P}(V)$, and
- $\nu_-(x, v)$ as the Hilbert norm on $T_{\varphi_\lambda^*(c_-)}\mathbf{P}(V^*)$ associated to $[\Theta^*]^\circ$ in $\mathbf{P}(V^*)$.

Now if $x = c(0)$ does not lie on a leaf of \mathcal{L} , then let $c(-t_-)$ (resp. $c(t_+)$) be the first intersection point between c and \mathcal{L} in the past (resp. future). Observe that there are uniform lower and upper bounds ε_- and ε_+ of the time period, for which a nonwandering geodesic crosses a connected component of $\mathbb{H}^2 \setminus \mathcal{L}$, i.e. $\varepsilon_- \leq t_+ + t_- \leq \varepsilon_+$. Define then $\nu_\pm(x, v)$ as the barycentric combination:

$$\frac{t_-}{t_+ + t_-} \nu_\pm(c(t_+)) + \frac{t_+}{t_+ + t_-} \nu_\pm(c(-t_-))$$

Recall that $C > 1$ is the uniform lower bound on the expansion of the Hilbert metrics (see (6) in Section 8.1 for the definition of C). Let N be the smallest integer such that $C^N > 2$. It follows that the norm $\nu_+(c(t), c'(t))$ is at least doubled and $\nu_-(c(t), c'(t))$ divided by 2 when c crosses at least $2N$ leaves of \mathcal{L} . Moreover, this surely happens when one travels along c for a time period $T = 2N\varepsilon_+$, and therefore item (ii) of Definition 5.1 is satisfied (see also Remark 5.2).

The proof of our main Theorem 1.1 is now complete.

9. EXTENSION OF NEW REPRESENTATIONS TO $\mathrm{PSL}(2, \mathbb{Z})$

In the Sections 6 and 8, we build a representation $\rho_\Theta^\lambda : \Gamma_o \rightarrow \mathcal{H}$ for every marked box $[\Theta]$ and every $\lambda = (\varepsilon, \delta) \in \mathbb{R}^2$, and prove that if $\lambda \in \mathcal{R}^\circ$, then ρ_Θ^λ is Anosov. In other words, we exhibit a subspace of $\mathrm{Rep}(\Gamma_o, \mathcal{H})$ which is made of Anosov representations and the boundary of which contains the restrictions to Γ_o of the Schwartz representations.

Hence, we can ask the following natural question:

When does the representation $\rho_\Theta^\lambda : \Gamma_o \rightarrow \mathcal{H}$ extend to a representation $\bar{\rho}_\Theta^\lambda : \Gamma \rightarrow \mathcal{G}$?

A main ingredient required for this extension is to find the image of the involution I : this image should be a polarity (see Remark 3.1), and since we know the images of R_* and IR_*I under ρ_Θ^λ , the problem of finding the image of I reduces to:

Find a polarity \mathcal{P} such that $\mathcal{B}_\Theta^\lambda = \mathcal{P}\mathcal{A}_\Theta^\lambda\mathcal{P}$.

Recall the proof of Lemma 6.7: The projective transformation $\mathcal{A}_\Theta^\lambda$ does not depend on λ and it corresponds to the matrix A_Θ computed in Lemma 4.5. The projective transformation $\mathcal{B}_\Theta^\lambda$ is exactly $\Sigma_\lambda^{-1} \mathcal{B}_\Theta^0 \Sigma_\lambda$ and $\mathcal{B}_\Theta^0 = \mathcal{D}_\Theta^0 \mathcal{A}_\Theta^0 \mathcal{D}_\Theta^0$, where \mathcal{D}_Θ^0 corresponds to the matrix D_Θ in Lemma 4.5, and for the Θ -basis of V , the transformation \mathcal{B}_Θ^0 corresponds to the matrix:

$$B_\Theta^0 := D_\Theta^{-1} {}^t(A_\Theta)^{-1} D_\Theta$$

Hence, $\mathcal{B}_\Theta^\lambda$ is represented by the matrix:

$$B_\Theta^\lambda := \Sigma_\lambda^{-1} D_\Theta^{-1} {}^t(A_\Theta)^{-1} D_\Theta \Sigma_\lambda$$

Now, the problem is to find an invertible symmetric matrix S such that:

$$S^{-1} {}^t(A_\Theta)^{-1} S = B_\Theta^\lambda.$$

When Θ is special, the solution is easy: In this case, since D_Θ is the identity matrix, we simply let $S = \Sigma_\lambda$.

From now on, let us assume that Θ is not special. In the appendix, we show through a computation that the existence of a non-zero symmetric matrix S satisfying the equation

$${}^t(A_\Theta)^{-1}S = SB_\Theta^\lambda$$

is equivalent to:

$$(7) \quad \det(\text{Id}_3 - A_\Theta B_\Theta^\lambda) = 0$$

and by another computation, Equation (7) holds if and only if $h(\varepsilon, \delta) = 0$, where

$$\begin{aligned} h(\varepsilon, \delta) = & (\zeta_t^2 + \zeta_b^2 - 2\zeta_t^2\zeta_b^2) \sinh(\delta) \cosh(\varepsilon) (2 \cosh(\delta) \cosh(\varepsilon) - \sinh(\varepsilon)) \\ & - \zeta_t\zeta_b(\zeta_t^2 - \zeta_b^2) \sinh(\varepsilon) (\cosh(\delta) \cosh(\varepsilon) - \sinh(\varepsilon) - 1). \end{aligned}$$

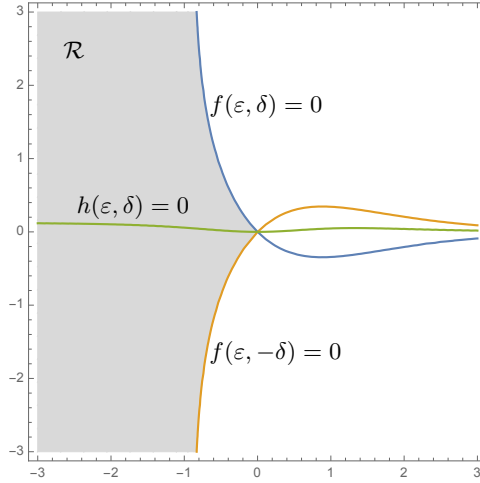


FIGURE 16. The equation $h(\varepsilon, \delta) = 0$ is drawn in green.

Let $\mathcal{C} = \{(\varepsilon, \delta) \in \mathbb{R}^2 \mid h(\varepsilon, \delta) = 0\}$ (see Figure 16). Since the invertibility of the matrix S is an open condition, there exists an open neighborhood \mathcal{U} of $(0, 0)$ in \mathbb{R}^2 such that for every $\lambda \in \mathcal{C} \cap \mathcal{U}$, the representations ρ_Θ^λ extends to a representation $\bar{\rho}_\Theta^\lambda : \Gamma \rightarrow \mathcal{G}$. Moreover, the following computation

$$\frac{\partial f}{\partial \delta}(0, 0) = -1 \quad \text{and} \quad \frac{\partial h}{\partial \delta}(0, 0) = 2(\zeta_t^2 + \zeta_b^2 - 2\zeta_t^2\zeta_b^2) \neq 0$$

and the implicit function theorem tell us that there are a neighborhood $V_\varepsilon \times V_\delta$ of $(0, 0)$ and two functions $\delta_f : V_\varepsilon \rightarrow \mathbb{R}$ and $\delta_h : V_\varepsilon \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} \{(\varepsilon, \delta_f(\varepsilon)) \mid \varepsilon \in V_\varepsilon\} &= \{(\varepsilon, \delta) \in V_\varepsilon \times V_\delta \mid f(\varepsilon, \delta) = 0\} \\ \{(\varepsilon, \delta_h(\varepsilon)) \mid \varepsilon \in V_\varepsilon\} &= \{(\varepsilon, \delta) \in V_\varepsilon \times V_\delta \mid h(\varepsilon, \delta) = 0\} \end{aligned}$$

Also, another simple computation

$$\frac{d\delta_f}{d\varepsilon}(0) = -1 \quad \text{and} \quad \frac{d\delta_h}{d\varepsilon}(0) = 0$$

shows that there exists an interval $\mathcal{V} :=]\varepsilon_0, 0] \subset V_\varepsilon$ such that:

$$(\varepsilon, \delta_h(\varepsilon)) \in \mathcal{R} \quad \text{for all } \varepsilon \in \mathcal{V} \quad \text{and} \quad (\varepsilon, \delta_h(\varepsilon)) \in \mathcal{R}^\circ \quad \text{for all } \varepsilon \in \mathcal{V}^\circ$$

Therefore, if we let $\lambda = (\varepsilon, \delta_h(\varepsilon))$ for every $\varepsilon \in \mathcal{V}$, then the representation ρ_Θ^λ extends naturally to a representation $\bar{\rho}_\Theta^\lambda : \Gamma \rightarrow \mathcal{G}$ when $\varepsilon \in \mathcal{V}$, it is Anosov when $\varepsilon \in \mathcal{V}^\circ$, and it is the restriction of the Schwartz representation when $\varepsilon = 0$.

Finally, we finish the proof of the main Theorem 1.2.

APPENDIX

A matrix A in $\mathrm{GL}(3, \mathbb{R})$ is a *rotation of angle θ* if there exists a matrix Q in $\mathrm{GL}(3, \mathbb{R})$ such that $Q^{-1}AQ = \mu R_\theta$, where $\mu \neq 0$ and

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Lemma 9.1. Let A be a rotation of angle θ . Assume that $0 < \theta < \pi$ and $B = G^{-1}A^{-1}G$ for some $G \in \mathrm{GL}(3, \mathbb{R})$. Then $\det(\mathrm{Id}_3 - AB) = 0$ if and only if there exists a symmetric matrix $S \neq 0$ such that $SB = {}^tA^{-1}S$.

Proof. By the assumption, there is a matrix Q in $\mathrm{GL}(3, \mathbb{R})$ such that $Q^{-1}AQ = \mu R_\theta$, and so ${}^tQ{}^tA^{-1}{}^tQ^{-1} = \mu^{-1}R_\theta$. This implies that:

$$\begin{aligned} SB = {}^tA^{-1}S &\Leftrightarrow SG^{-1}{}^tA^{-1}G = {}^tA^{-1}S \\ &\Leftrightarrow ({}^tQSQ)({}^tQGGQ)^{-1}R_\theta({}^tQGGQ) = R_\theta({}^tQSQ) \end{aligned}$$

As a consequence, there is a non-zero symmetric matrix S satisfying $SB = {}^tA^{-1}S$ if and only if there exists a symmetric matrix $P \neq 0$ such that:

$$P({}^tQGGQ)^{-1}R_\theta = R_\theta P({}^tQGGQ)^{-1}$$

It follows that R_θ commutes with $P({}^tQGGQ)^{-1}$, and therefore

$$P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & -\gamma \\ 0 & \gamma & \beta \end{pmatrix} ({}^tQGGQ) \quad \text{for some } \alpha, \beta, \gamma \in \mathbb{R}.$$

If $U = (u_{ij})_{i,j=1,2,3}$ denotes tQGGQ , then we can write the equation $P - {}^tP = 0$ as follows:

$$(8) \quad \begin{pmatrix} -u_{12} & u_{21} & -u_{31} \\ -u_{13} & u_{31} & u_{21} \\ 0 & u_{32} - u_{23} & u_{22} + u_{33} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let M be the left 3×3 matrix of Equation (8). Then by a simple computation, we have:

$$\begin{aligned} 2 \sin(\theta)(1 - \cos(\theta)) \cdot \det(M) &= \det(U) \cdot \det(\mathrm{Id}_3 - R_\theta U^{-1} R_\theta U) \\ &= \det(U) \cdot \det(\mathrm{Id}_3 - AG^{-1}{}^tA^{-1}G) \end{aligned}$$

In the last step, we use the fact that:

$$AG^{-1}{}^tA^{-1}G = QR_\theta U^{-1} R_\theta U Q^{-1}$$

Finally, $\det(M) = 0$ if and only if $\det(\mathrm{Id}_3 - AB) = 0$, which completes the proof. \square

Remark 9.2. One implication in Lemma 9.1 is easier to prove without computation. If $B = S^{-1}{}^tA^{-1}S$ with S an invertible symmetric matrix, then:

$$\begin{aligned} \text{Id}_3 - AB &= \text{Id}_3 - AS^{-1}{}^tA^{-1}S \\ &= AS^{-1}(SA^{-1} - {}^tA^{-1}S) \\ &= AS^{-1}(SA^{-1} - {}^t(SA^{-1})) \quad (S \text{ is symmetric}) \end{aligned}$$

Notice that $SA^{-1} - {}^t(SA^{-1})$ is an anti-symmetric 3×3 matrix, which implies that

$$\det(\text{Id}_3 - AB) = 0.$$

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